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RECTIFIABLE CURVES AND THE WEIERSTRASS INTEGRAL

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Recent research on surface area theory and the calculus of variations has drawn attention to the theory of rectifiable curves and the Weierstrass integral. Indeed, most basic theorems for curves have their two-dimensional counterpart, although the proofs and the underlying concepts for the two-dimensional case lie much deeper than for curves and show unexpected connections with topology, measure theory, and functional analysis.

The theory for rectifiable curves is presented sometimes in courses and books on real functions or on integration. The present brief exposition intends to emphasize those parts which are of greater interest because of their recent extensions to surfaces.

1. Parametric curves. We should first define the terms we need: parametric curves, continuous curves, rectifiable curves. For parametric curves we shall assume the following definition, which is adequate for the present exposition:

A system of equations

$$C:\ x_1 = x_1(u),\ x_2 = x_2(u),\ \cdots,\ x_N = x_N(u),\quad a \leqq u \leqq b, \tag{1.1}$$

where $x_i(u)$ are real functions of u is said to be a *parametric curve* C in the real Euclidean x-space E_N, $x=(x_1, \cdots, x_N)$, and $-\infty < a < b < \infty$.
Thus in vectorial notation we have

$$C:\ x = x(u),\qquad a \leqq u \leqq b, \tag{1.1}$$

and C may be thought of as a *mapping* from $[a, b]$ into E_N. The curve C is said to be *continuous* if the functions $x_i(u)$, $i=1, \cdots, N$, are all continuous in $[a, b]$. We shall also define the *graph* $[C]$ of C as a concept apart from the curve C. By graph $[C]$ we mean, as usual, the set of all points $x \in E_N$ "covered," (or "occupied," or "travelled") by C in E_N, *i.e.*, $[C] = [x \in E_N;\ x = f(u),\ a \leqq u \leqq b]$. For every $u \in [a, b]$, $x(u)$ is said to be the *image* of u on C. It is often convenient to assume on $[a, b]$ the natural order from a to b. Then the curve C is said to be *oriented*. A point $x \in [C]$ may be covered once, finitely many times, infinitely many times by C; *i.e.*, a point $x \in [C]$ may be the image of one, finitely many, or infinitely many points $u \in [a, b]$. The examples (Fig. 1) for $N=2$ may illustrate the situation: $C_1: x_1 = t,\ x_2 = t^2,\ -1 \leqq t \leqq 1$; $C_2: x_1 = \cos 2u,\ x_2 = \cos 2u \tan u,\ -\pi/3 \leqq u \leqq \pi/3$; $C_3: x_1 = x_1(u),\ x_2 = x_2(u),\ 0 \leqq u \leqq 1$, where $x_1(u) = u \cos 2\pi u^{-1}$,

$x_2(u) = u \sin 2\pi u^{-1}$ if $0 < u \leqq 1$, $x_1 = x_2 = 0$ if $u = 0$; C_4: $x_1 = |u|$, $x_2 = x_2(u)$, $-1 \leqq u \leqq 1$, where $x_2(u) = |u| \sin 2\pi u^{-1}$ if $-1 \leqq u \leqq 1$, $u \neq 0$, $x_2 = 0$ if $u = 0$.

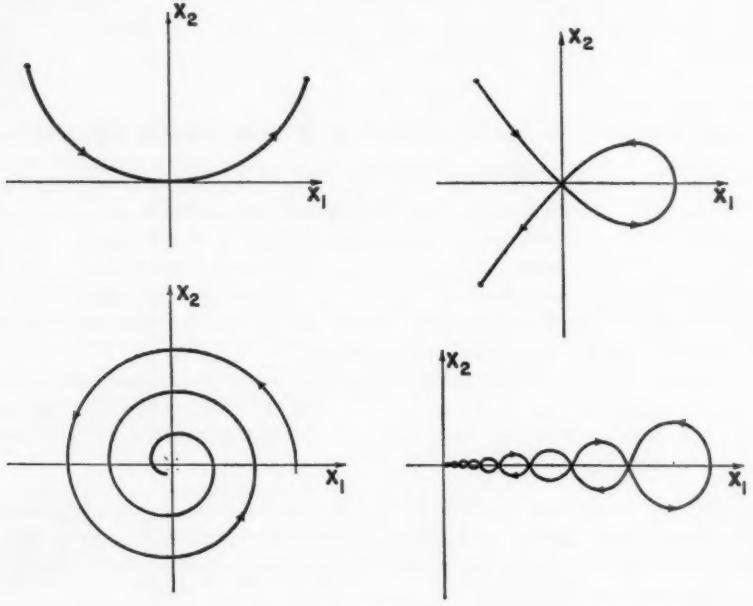

FIG. 1.

It may well occur that completely different curves C have the same graph. For instance, the curves D_k: $x_1 = \sin^2 ku$, $x_2 = 0$, $0 \leqq u \leqq \pi/2$, $k = 1, 2, \cdots$, all cover the segment $s = [D_k] = [0 \leqq x_1 \leqq 1, x_2 = 0]$ joining the points (0, 0), (1, 0) in E_2, but D_1 covers s just once, D_2 covers s twice (back and forth), D_3 covers s three times, *etc.* The curves $D_1, D_2, D_3, \cdots$ are certainly different curves (and indeed they will have different lengths 1, 2, $\cdots$), but have the same graph. Analogously, the curves F_k: $x_1 = \cos ku$, $x_2 = \sin ku$, $0 \leqq u \leqq 2\pi$, all cover the circle $t = [x_1^2 + x_2^2 = 1]$, but F_1 covers t once ((1, 0) is covered twice), F_2 covers t twice, *etc.* The curves $F_1, F_2, \cdots$ are different curves but they have the same graph. The graph $[C]$ of a curve C may be a set of a very complicated nature (a characterization will be mentioned in Section 7). The graph of a curve C may well be a set of positive measure, and may have interior points; it may be a square, a cube, *etc.* (F. Osgood, G. Peano) (see *e.g.*, E. W. Hobson, *Functions of a real variable*, I, pp. 451–458).

There are situations where our intuition associates to two different sets of equations (1.1) the same entity. For instance, this occurs for the two curves in E_2

$$C: x_1 = u, x_2 = 0, 0 \leqq u \leqq 1, \qquad C': x_1 = \sin v, x_2 = 0, 0 \leqq v \leqq \pi/2.$$

More generally, suppose $u=\phi(v)$, $c\leqq v\leqq d$, to be a continuous, always-increasing function with $a=\phi(c)$, $b=\phi(d)$. Thus ϕ maps $[c, d]$ onto $[a, b]$, each value $a\leqq u\leqq b$ is taken by $\phi(v)$ only once, and the inverse function $v=\phi^{-1}(u)$, $a\leqq u\leqq b$, exists, is continuous, and maps $[a, b]$ onto $[c, d]$. Then any two curves as

$$C: x = x(u),\ a \leqq u \leqq b, \quad C': x = x[\phi(v)],\ c \leqq v \leqq d,$$

are felt to "represent" the same entity. For instance, in the example above $\phi(v)=\sin v$, $0\leqq v\leqq\pi/2$. In Section 8 we shall introduce concepts of equivalence (Lebesgue, Fréchet). Thus C, C' will be denoted as Lebesgue equivalent. Classes of Lebesgue [Fréchet] equivalent curves will be said to define the same Lebesgue [Fréchet] curve.

Finally, it should be pointed out that "curves" as $x_1=t$, $x_2=t^2$, $-\infty<t<+\infty$ (a parabola), $x_1=(1-t)\cos t^{-1}$, $x_2=(1-t)\sin t^{-1}$, $0<t\leqq 1$ (a spiral), $x_1=t$, $x_2=(a^2+t^2)^{1/2}>0$, $0\leqq t<+\infty$ (half-hyperbola), are not parametric continuous curves according to the definition above since the parameter t does not range on an interval both closed and finite. A slightly more general concept of curve should be considered: $C: x=x(u)$, $u\in G$, where G is any set of real numbers. Some of the main theorems can be extended to this situation without effort (see, *e.g.*, S. Saks, *Theory of the integral*, Warsawa, 1937, pp. 121–125).

2. The Jordan length. By norm $|x|$ of $x\in E_N$ we mean, as usual, $|x|=(x_1^2+\cdots+x_N^2)^{1/2}\geqq 0$, and thus for any two points x, $y\in E_N$, $|x-y|$ is the Euclidean distance

$$|x-y| = [(x_1-y_1)^2+\cdots+(x_N-y_N)^2]^{1/2} \geqq 0. \tag{2.1}$$

Thus, for $N=1$, $|x|$ is the absolute value of x, and $|x-y|$ the usual distance on the real axis.

For curves $C: x=x(u)$, $a\leqq u\leqq b$, for which the functions $x_1(u), \cdots, x_N(u)$, are continuous with their first derivatives $x_1'(u), \cdots, x_N'(u)$, the length of C is often assumed, by definition, to be the numerical value of the integral (length integral)

$$\int_a^b |x'(u)|\,du = \int_a^b (x'^2+\cdots+x_N'^2)^{1/2}du. \tag{2.2}$$

This definition is certainly adequate under the hypotheses above, but it is not adequate under somewhat weaker assumptions, as the following examples show.

Let $\phi(u)$, $0\leqq u\leqq 1$, denote the well-known monotone nondecreasing continuous function with $\phi(0)=0$, $\phi(1)=1$, which is constant on each complementary interval $I\subset[0, 1]$ of the ternary Cantor set E in $[0, 1]$, where the countable sum of the lengths of the intervals I is exactly 1 [see, *e.g.*, E. W. Hobson, *Functions of a real variable*, I, p. 368]. Thus $\phi'(u)=0$ at all interior points of the intervals I and hence almost everywhere (a.e.) in $[0, 1]$. The curve (Fig. 2) $C: x_1=u$, $x_2=\phi(u)$, $0\leqq u\leqq 1$, in E_2, is a continuous curve of end points (0, 0), (1, 1). The chord through the end points of C has length $\sqrt{2}$, and we should

expect for the length of C a value $\geqq\sqrt{2}$. The formula (2.2) gives the value $\int_0^1(1+0)^{1/2}du=1$. (The Jordan length of C as defined below will be 2).

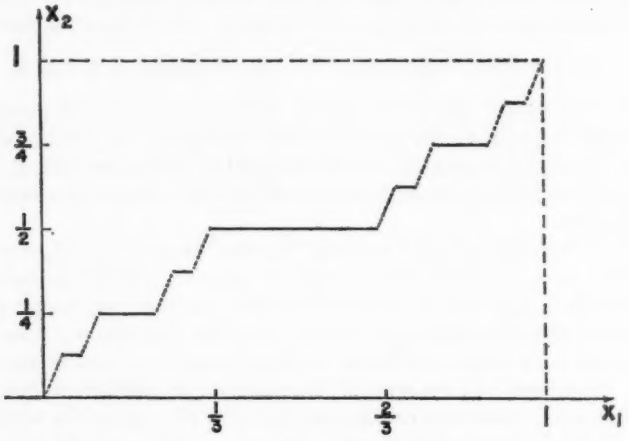

FIG. 2.

As another example consider the continuous curve $D: x_1=\phi(u)$, $x_2=0$, $0\leqq u\leqq 1$, in E_2. Essentially D is the segment s of end points (0, 0), (1, 0) (according to Section 8, D is actually Fréchet equivalent to the curve $D_0: x_1=u$, $x_2=0$, $0\leqq u\leqq 1$). Thus we should expect for the length of D the value 1 (or a value $\geqq 1$). The formula (2.2) gives the value $\int_0^1(0+0)^{1/2}du=0$.

These examples show that formula (2.2) is not adequate under hypotheses weaker than the one mentioned at the beginning of this section. C. Jordan, in 1884, proposed and studied a definition of length which is completely adequate for all parametric curves (continuous as well as discontinuous).

By *Jordan length* $l(C)$ of the curve C is meant the supremum of the elementary lengths of the inscribed polygonal lines. More precisely, if $D=[a=u_0<u_1<\cdots<u_n=b]$ denotes any finite subdivision of $[a, b]$ into consecutive subintervals, the definition of $l(C)$ is

$$l(C)=l_1(C)=\operatorname*{Sup}_D\sum_{i=1}^{n}\left|x(u_i)-x(u_{i-1})\right|. \tag{2.3}$$

For $N=1$, C is any given real function of the real variable u in $[a, b]$, and $l(C)$ is its total variation. A curve C in E_N is said to be *rectifiable* if $l(C)<+\infty$.

If $I=[\alpha, \beta]$ is any subinterval of $[a, b]$, by $x(I)$ we denote the set of all points $x\in E_N$ with $x=x(u)$ for some $u\in[\alpha, \beta]$. By *oscillation* of $x(u)$ in $[\alpha, \beta]$ or osc $(x; \alpha, \beta)$ we denote, as usual, the diameter of the set $x(I)$, *i.e.*, osc $(x; \alpha, \beta)$ $=\operatorname{diam} x(I)=\operatorname{Sup}\left|x(u')-x(u)\right|$, where the supremum is taken over all $u, u'\in I$. Thus $\left|x(\alpha)-x(\beta)\right|\leqq$ osc $(x; \alpha, \beta)$ for every $I=[\alpha, \beta]$.

An alternative definition of Jordan length is the following one

$$l_2(C) = \operatorname*{Sup}_D \sum_{i=1}^{n} \operatorname{osc}(x; u_{i-1}, u_i). \tag{2.4}$$

(2.i) *For every continuous curve C we have $l_1(C) = l_2(C)$.*

The proof is the same as given in a previous article (this MONTHLY, vol. 65, 1958, pp. 317–332, Section 1, no. 1) for the total variation of a real function of one real variable. We shall refer to this article by I.

Given, as above, a curve C: $x = x(u)$, $a \leqq u \leqq b$, and a subdivision $D = [a = u_0 < u_1 < \cdots < u_n = b]$ of $[a, b]$, we shall denote by d and δ two possible *norms* of D with respect to C, namely $d = \max(u_i - u_{i-1})$, and $\delta = \max \operatorname{osc}(x; u_{i-1}, u_i)$, where in both the maximum is taken for i ranging over $i = 1, 2, \cdots, n$. If C is a continuous curve, then $d \to 0$ implies $\delta \to 0$, but the converse is not true, as it is clear when $x(u)$ is constant on some subinterval of $[a, b]$. For continuous curves C the Jordan length is actually a limit of the sums (2.3), (2.4):

(2.ii) *For every continuous curve C we have*

$$l_1(C) = l_2(C) = \lim_{\delta \to 0} \sum_{i=1}^{n} |x(u_i) - x(u_{i-1})| = \lim_{\delta \to 0} \sum_{i=1}^{n} \operatorname{osc}(x; u_{i-1}, u_i),$$

and the same holds as $d \to 0$.

The proof is the same as given in I for the total variation of a function of one real variable.

3. The Jordan theorem. We shall denote by V_r the total variation in $[a, b]$ of the function $x_r(u)$, $r = 1, \cdots, N$. Thus $x_r(u)$, $a \leqq u \leqq b$, is a function of bounded variation (BV) if and only if $V_r < +\infty$. Absolute continuous functions will be denoted as usual by AC functions (cf. I).

(3.i) (C. Jordan, 1884) *For every parametric curve C we have*

$$V_r \leqq l(C) \leqq V_1 + \cdots + V_N, \qquad r = 1, \cdots, N. \tag{3.1}$$

Hence C is rectifiable if and only if all functions $x_r(u)$, $r = 1, \cdots, N$, are BV in $[a, b]$.

Proof. For every two points $x, y \in E_N$ we have, from (2.1),

$$|x_r - y_r| \leqq |x - y| \leqq |x_1 - y_1| + \cdots + |x_N - y_N|, r = 1, \cdots, N.$$

Hence, for every subdivision D of $[a, b]$ we have

$$\sum_{i=1}^{N} |x_r(u_i) - x_r(u_{i-1})| \leqq \sum_{i=1}^{n} |x(u_i) - x(u_{i-1})| \leqq \sum_{r=1}^{N} \sum_{i=1}^{n} |x_r(u_i) - x_r(u_{i-1})|,$$

$r = 1, \cdots, N$. By taking the supremum of each member we deduce relation

(3.1). This relation implies that $l(C)<+\infty$ if and only if all numbers V_r, $r=1, \cdots, N$, are finite.

Theorem (3.i) was proved by Jordan in 1884 in just the same form, *i.e.*, for continuous as well as discontinuous curves. The Jordan length has a number of formal properties in common with the total variation of a real function $f(u)$ $[N=1]$, namely all those properties of the latter which involve only vectorial properties of the real system. One of these properties is the following one:

(3.ii) *If* $I=[a, b]$ *and* $I_1, \cdots, I_m$, *is a finite subdivision of* I *into subintervals, and we denote by* $C, C_1, \cdots, C_m$, *the curves defined by the vector function* $x(u)$ *on* $I, I_1, \cdots, I_m$, *respectively, then* $l(C)=l(C_1)+\cdots+l(C_m)$.

4. Lower semicontinuity of the Jordan length. This is the basic property of the Jordan length and holds for continuous as well as discontinuous parametric curves. This property is well expressed by the following theorem:

(4.i) *If* $C: x=x(u)$, $a\leqq u\leqq b$, $C_n: x=x_n(u)$, $a\leqq u\leqq b$, $n=1, 2, \cdots$, *are given curves and* $x(u)=\lim x_n(u)$ *as* $n\to+\infty$ *for every* $u\in[a, b]$, *then*

$$l(C)\leqq \liminf_{n\to\infty} l(C_n). \tag{4.1}$$

Proof. Given $\epsilon>0$ there is a subdivision D of $[a, b]$ such that

$$\sum_{i=1}^{N} \left| x(u_i)-x(u_{i-1})\right| > l(C)-\epsilon, \text{ or } 1/\epsilon,$$

according as $l(C)<+\infty$, or $l(C)=+\infty$. Since $x_n(u_i)\to x(u_i)$ as $n\to\infty$, $i=0, 1, \cdots, N$, there is an n_i such that $\left|x_n(u_i)-x(u_i)\right|<\epsilon/2N$ for all $n\geqq n_i$, $i=0, 1, \cdots, N$, and hence, also, for $n\geqq n_0=\max n_i$. For $n\geqq n_0$ we have also $\left|x_n(u_i)-x_n(u_{i-1})\right| > \left|x(u_i)-x(u_{i-1})\right|-\epsilon/N$, and finally $l(C)-\epsilon$, or $1/\epsilon < \sum_{i=1}^{N}\left[\left|x_n(u_i)-x_n(u_{i-1})\right|+\epsilon/N\right]\leqq l(C_n)+\epsilon$ for all $n\geq n_0$. This implies (4.1). This proof is the same as in I, (7.i).

A variant of (4.i) for continuous curves is

(4.ii) *If* $C: x=x(u)$, $a\leqq u\leqq b$, $C_n: x=c_n(u)$, $a\leqq u\leqq b$, $n=1, 2, \cdots$, *are continuous curves and* $x(u)=\lim x_n(u)$ *as* $n\to\infty$ *for at least all* u *of a set which is everywhere dense in* $[a, b]$, *then* (4.1) *holds.*

Remark 1. In neither of the statements (4.i), (4.ii) uniform convergence is required, and, in (4.i), the continuity of the curves is not required.

Remark 2. Examples show that equality sign does not hold necessarily in (4.1). For instance, let $C: x_1=u, x_2=0, 0\leqq u\leqq 1$, and $C_n: x_1=u, x_2=n^{-1}\sin 2\pi n^s u$, $0\leqq u\leqq 1$, where n, n^s are all integers. Then $l(C)=1$, and, by (3.1), also $4n^s\left[(4n^s)^{-2}+n^{-2}\right]^{1/2}\leqq l(C_n)\leqq 4n^s\left[(4n^s)^{-1}+n^{-1}\right]$.

If $n=m^2$, $s=1/2$, $m=1, 2, \cdots$, we have $l(C)=1=\lim l(C_m)$ as $m\to+\infty$. If $s=1$, $n=1, 2, \cdots$, we have $l(C)=1<17^{1/2}\leqq\lim l(C_n)\leqq\limsup l(C_n)\leqq 5$. If $s=2$, $n=1, 2, \cdots$, we have $l(C)=1<\lim l(C_n)=+\infty$.

Remark 3. A continuous curve $C: x=x(u)$, $a \leqq u \leqq b$, is said to be a *polygonal line* (and $x(u)$ quasilinear in $[a, b]$), if there is some finite subdivision D of $[a, b]$ into parts on each of which $x(u)$ is linear or constant. Then each part is mapped by $x(u)$ onto a segment σ and the elementary length $l_e(C)$ is the sum of the lengths of these segments. Obviously the Jordan length $l(C)$ coincides with the elementary length for every polygonal line C.

Now let us consider any continuous curve $C: x=x(u)$, $a \leqq u \leqq b$, and the class γ of all sequences $\gamma_n = [x=x_n(u),\ a \leqq u \leqq b]$, $n=1, 2, \cdots$, of polygonal lines with $x_n(u) \to x(u)$ uniformly in $[a, b]$ as $n \to \infty$. By (4.i) we have

$$l(C) \leqq \liminf_{n\to\infty} l(\gamma_n) \equiv \liminf_{n\to\infty} l_e(\gamma_n),$$

while, by the definition of Jordan length (Section 2), we know that there is some sequence with $l_e(C_n) \to l(C)$ as $n \to \infty$. This remark suggests the following alternative definition of Jordan length

$$l(C) = \operatorname*{Inf}_{\gamma} \left[\liminf_{n\to\infty} l_e(\gamma_n)\right].$$

5. The function $s(u)$. For every $a \leqq u \leqq b$, let $s(u)$ denote the length of the curve defined by (1.1) on the interval $[a, u]$, and by $v_r(u)$ the total variation of $x_r(u)$ in $[a, u]$. Then by (3.ii) and the corresponding theorem for total variations (cf. I, Section 1, Remark 1), we deduce that, for each interval $[\alpha, \beta]$ of $[a, b]$, the differences $s(\beta)-s(\alpha)$, $v_r(\beta)-v_r(\alpha)$ are the length of the curve defined by (1.1) on $[\alpha, \beta]$, and the total variation of $x_r(u)$ on $[\alpha, \beta]$, $r=1, \cdots, N$, respectively. The functions $s(u)$, $v_r(u)$ are monotone nondecreasing in $[a, b]$, and $s(a) = v_r(a) = 0$, $s(b) = l(C)$, $v_r(b) = V_r$, $r=1, \cdots, N$.

(5.i) *For every curve C and any subinterval $[\alpha, \beta]$ of $[a, b]$ we have*

$$v_r(\beta) - v_r(\alpha) \leqq s(\beta) - s(\alpha) \leqq \sum_{r=1}^{N} [v_r(\beta) - v_r(\alpha)], \qquad r = 1, \cdots, N,$$

$$|x_r(\beta) - x_r(\alpha)| \leqq |x(\beta) - x(\alpha)| \leqq s(\beta) - s(\alpha), \qquad r = 1, \cdots, N.$$

This statement is a consequence of the definitions and of (3.i).

(5.ii) *For every continuous rectifiable curve C the functions $s(u)$, $v_r(u)$ $a \leqq u \leqq b$, are continuous in $[a, b]$, and $s(a) = v_r(a) = 0$, $s(b) = l(C) < +\infty$, $v_r(b) = V_r < +\infty$, $r=1, \cdots, N$.*

This statement is a consequence of (5.i) and of the same property for the total variations $v_r(u)$ (cf. I, Section 6).

(5.iii) *For every continuous rectifiable curve C, the function $s(u)$ is AC if and only if all functions $v_r(u)$, $r=1, \cdots, N$, are AC, and hence if and only if all functions $x_r(u)$, $r=1, \cdots, N$, are AC.*

The first part of this statement is a consequence of (5.i), the second part of the analogous statement for the total variation (cf. I, Section 6).

6. The Tonelli theorems.

(6.i) (L. Tonelli, 1908–12) *For every continuous rectifiable curve* $C: x = x(u)$, $a \leqq u \leqq b$, $x(u) = [x_r(u), r = 1, \cdots, N]$, *we have*

$$l(C) \geqq \int_a^b |\, x'(u)\,|\, du, \tag{6.1}$$

and the $=$ *sign holds if and only if all functions* $x_r(u)$, $r = 1, \cdots, N$, *are* AC.

Proof. By (3.i) all functions $x_r(u)$ are BV in $[a, b]$ and, by a theorem of Lebesgue [cf. I, Section 11], the derivatives $x_r'(u)$ exist almost everywhere (a.e.) in $[a, b]$ and are L-integrable. Also, the nondecreasing function $s(u)$, $a \leqq u \leqq b$, $s(a) = 0$, $s(b) = l(C)$ has derivative $s'(u)$ a.e. in $[a, b]$, $s'(u)$ is L-integrable in $[a, b]$, and, by (5.i), we may deduce that $|\, x'(u)\,| \leqq s'(u)$ a.e. in $[a, b]$. By the same Lebesgue theorem we have now

$$l(C) = s(b) - s(a) \geqq \int_a^b s'(u)du \geqq \int_a^b |\, x'(u)\,|\, du,$$

and (6.1) is proved. Suppose that the $=$ sign holds in (6.1). Then $s(b) - s(a) = \int_a^b s'(u)du$, while, for every $a \leqq u \leqq b$, by (6.1) we have

$$s(u) - s(a) \geqq \int_a^u s'(u)du, \qquad s(b) - s(u) \geqq \int_u^b s'(u)du. \tag{6.2}$$

By addition and comparison we have $+\infty > s(b) - s(a) \geqq \int_a^b s'(u)du = s(b) - s(a)$ and thus the $=$ sign must hold in both relations (6.2) for all $a \leqq u \leqq b$. This implies that $s(u) = s(a) + \int_a^u s'(u)du$ for all u, *i.e.*, $s(u)$ is an integral function, and thus $s(u)$ is AC. By (5.iii) all $x_r(u)$, $r = 1, \cdots, N$, are AC in $[a, b]$. Suppose finally that all functions $x_r(u)$, $r = 1, \cdots, N$, are AC in $[a, b]$. Then by (5.iii) and for every subdivision D of $[a, b]$ we have

$$\begin{aligned} \sum_{i=1}^{n} |\, x_r(u_i) - x_r(u_{i-1})\,| &= \sum_{i=1}^{n} \Big[\sum_{r=1}^{N} \Big(\int_{u_{i-1}}^{u_i} x_r'(u)du \Big)^2 \Big]^{1/2} \\ &\leqq \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} \Big(\sum_{r=1}^{n} x_r'^{\,2}(u) \Big)^{1/2} du = \int_a^b |\, x'(u)\,|\, du. \end{aligned} \tag{6.3}$$

By taking the supremum of the first member for all D we have $l(C) \leqq \int_a^b |\, x'(u)\,|\, du$. By comparison with (6.1) we conclude that the $=$ sign holds in (6.1).

Remark. The central inequality in (6.3) has been obtained by applying a known inequality for real numbers, often given in either form

$$[(\sum a_i)^2 + (\sum b_i)^2 + (\sum c_i)^2]^{1/2} \leqq \sum (a_i^2 + b_i^2 + c_i^2)^{1/2},$$

$$\left(\left[\int_I f du\right]^2 + \left[\int_I g du\right]^2 + \left[\int_I h du\right]^2\right)^{1/2} \leqq \int_I (f^2 + g^2 + h^2)^{1/2} du.$$

See, *e.g.*, S. Saks, p. 171. The first one is just a form of the so-called triangle inequality. The various parts of the previous proof can be traced, with some simplifications, in the original papers of L. Tonelli, [*Acc. Sci. Torino*, 43, 1908, 783–800, and 47, 1912, 1067–1075].

(6.ii) (L. Tonelli, 1912). *For every continuous rectifiable curve C we have* $s'(u) = |x'(u)|$ *a.e. in* $[a, b]$.

A simple proof of this theorem is given in L. M. Graves, *Functions of real variables*, 1946, p. 213.

7. The Noebeling theorem. Since a parametric continuous curve is, by definition, a continuous mapping from a finite closed interval $a \leqq u \leqq b$ into E_N, the set $[C]$ has a number of properties, namely $[C]$ is bounded, and closed (thus compact), and connected (thus a continuum). In addition it is locally connected, even "uniformly locally connected." A set $M \subset E_N$ is said to have this property if, given $\epsilon > 0$, there is a $\delta > 0$ such that for any two points x, $y \in M$, $|x-y| < \delta$ there is some subcontinuum $m \subset M$, with x, $y \in m$, diam $m < \epsilon$. By a theorem of H. Hahn and Mazurkiewicz, a set M is the graph $[C]$ of a (parametric continuous) curve C if and only if M is bounded, closed, connected and uniformly locally connected. Such a set is also said to be a Peano space. A square in E_2, a cube in E_3, certainly satisfy these conditions and thus are the graphs of some continuous curve. In topology such a set is said to be "a continuous curve"; it is unfortunate that the use of this term clashes with the one of the present article, of differential geometry, of calculus of variations, and other parts of mathematics.

For each point $x \in [C]$ let us denote by $M(x)$ the number (finite or $+\infty$) of the disjoint points $u \in [a, b]$ with $x = f(u)$; *i.e.* $M(x)$, is the number of points u of $[a, b]$ of which x is the image under C. For each $x \in E_N - [C]$, let $M(x) = 0$. Then the *multiplicity function* $M(x)$, $0 \leqq M(x) \leqq +\infty$, is defined for all $x \in E_n$, and is nonnegative and integral valued. We shall denote by H^1 the 1-dimensional Hausdorff measure in E_N.

(7.i) (G. Noebeling, 1940) *For every continuous parametric curve C we have*

$$l(C) = \int_{E_N} M(x) dH^1.$$

This theorem connects the Jordan length of a continuous curve C with the concept of 1-dimensional Hausdorff measure of a set. If $\phi(x)$ is the characteristic function of the set $[C]$, *i.e.*, $\phi = 1$ for $x \in [C]$, $\phi = 0$ for $x \in E_N - [C]$, then

$$H^1([C]) = \int_{E_N} \phi(x) dH^1 \leqq \int_{E_N} M(x) dH^1 = l(C),$$

and it may well occur that $H^1([C]) < l(C)$, as, *e.g.*, for the curves D_k, F_k, $k \geqq 2$, of Section 1.

For every set $A \subset E_N$, Lebesgue measure, say $|A|$, and N-dimensional Hausdorff measure coincide, say $|A| = H^N(A)$. For $N > 1$, the measure $|[C]|$ of the graph $[C]$ of a continuous curve C in E_N is not necessarily zero [indeed, $[C]$ may be a cube in E_N, *etc.*]. Nevertheless, if C is rectifiable and $N > 1$, then $|[C]| = 0$ [see, *e.g.* L. Cesari, *Surface area*, 1956, p. 93]. (For the concept of measure H^1, which will not be used in the following, see also, *e.g.*, S. Saks, *loc. cit.*, p. 53).

8. Lebesgue and Fréchet equivalences. If $C: x = x(u)$, $a \leqq u \leqq b$, we may identify a and b, and then we must require $x(a) = x(b)$. Then C is said to be a *closed* curve, and $[a, b]$ becomes topologically equivalent to a circle, *i.e.*, a 1-sphere. Otherwise we say that C is *open*, and in this case a and b are not identified, and $[a, b]$ is a 1-cell. Since the concept of equivalence for closed curves requires a few more words than for open curves, we will suppose that all curves in this section are (parametric, continuous, and) open. Also, we shall suppose that they are all oriented.

A mapping $u = h(v)$, $v \in J$, $u \in I$, $h(c) = a$, $h(d) = b$, from the interval $J = [c \leqq v \leqq d]$ onto the interval $I = [a \leqq u \leqq b]$, is said to be a homeomorphism from J onto I if it is one-one and continuous together with its inverse, $v = h^{-1}(u)$. Given any two (oriented, open, continuous, parametric) curves

$$C: x = f(u),\ u \in I = [a, b], \qquad C_1: x = g(v),\ v \in J = [c, d],$$

in E_N, we say that C is Lebesgue-equivalent to C_1 (or L-equivalent to C_1) if there is a homeomorphism from J onto I, $u = h(v)$, $v \in J$, $u \in I$, such that $f[h(v)] = g(v)$ for all $v \in J$. It is easy to prove that L-equivalence has the standard symmetric, reflexive, transitive properties for an equivalence. Thus we may say, as usual, that two curves C, C_1 are L-equivalent. We have already given in Section 1 an example of two L-equivalent curves.

A more general concept of equivalence is the following one. Given any two curves C, C_1 as above, we say that C is Fréchet equivalent to C_1 (F-equivalent, $C \sim C_1$) provided, given $\epsilon > 0$, there exists a homeomorphism $u = h_\epsilon(v)$, $v \in J$, $u \in I$ (which may depend on ϵ), such that $|f[h_\epsilon(v)] - g(v)| < \epsilon$ for all $v \in J$. Again it is easy to prove that the F-equivalence has the same properties for an equivalence mentioned above, *i.e.*, 1. $C \sim C$; 2. $C \sim C_1$ implies $C_1 \sim C$; $C \sim C_1$, $C_1 \sim C_2$ implies $C \sim C_2$.

Obviously F-equivalent curves are L-equivalent, but the converse is not true, as the following example shows. Suppose (Fig. 3) C: $x_1 = u$, $x_2 = \cdots = x_N = 0$, $u \in I = [0, 1]$; C_1: $x_1 = g(u)$, $x_2 = \cdots = x_N = 0$, $u \in J = [0, 1]$, where $g(u) = 3u/2$ if $0 \leqq u \leqq 1/3$, $g(u) = 1/2$ if $1/3 \leqq u \leqq 2/3$, $g(u) = 1 + (3/2)(u - 1)$, if $2/3 \leqq u \leqq 1$. Let us prove that $C \sim C_1$. For every $n \geqq 4$ let $u = h_n(v)$, where $h_n(v) = 3v/2$ if $0 \leqq v \leqq (1/3) - (1/n)$; $h_n(v) = 1 + (3/2)(v - 1)$ if $(2/3) + (1/n) \leqq v \leqq 1$; $h_n(v) = (1/2) + 9(n+6)^{-1}(v - 1/2)$ if $(1/3) - (1/n) \leqq v \leqq (2/3) + (1/n)$. Let $h(v) = g(v)$,

$f(u)=u$. Then, by elementary computations, we have $|h_n(v)-h(v)|<3/n$ for all $0\leqq v\leqq 1$. Thus $|f[h_n(v)]-f[h(v)]|\to 0$ uniformly in $[0, 1]$, and this proves, since $f[h(v)]=g(v)$, that $C\sim C_1$.

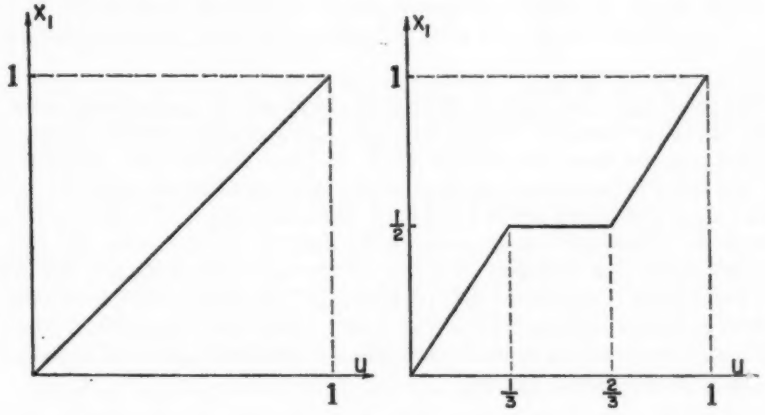

FIG. 3.

(8.i) *If C, C' are L-equivalent, or F-equivalent, then $l(C)=l(C')$, i.e., the Jordan length is invariant with respect to both L- and F-equivalences.*

Proof. For L-equivalence, observe that, if C: $x=f(u)$, $a\leqq u\leqq b$, C': $x=g(v)$, $c\leqq v\leqq d$, then $f[h(v)]=g(v)$, $f(u)=g[h^{-1}(u)]$ for some homeomorphism h, and any subdivision D of $[a, b]$ is mapped by h into a subdivision D' of $[c, d]$ (and conversely). Since the corresponding sums (2.3) are equal, we deduce $l(C)\leqq l(C')$ as well as $l(C')\leqq l(C)$. Thus $l(C)=l(C')$. For F-equivalence observe that for each $n=1, 2, \cdots$, there exists a homeomorphism h_n such that, if C_n: $x=f_n(v)\equiv f[h_n(v)]$, $c\leqq v\leqq d$, then $|f_n(v)-g(v)|=|f[h_n(v)]-g(v)|<1/n$ for all v, *i.e.*, $f_n(v)\to g(v)$ as $n\to\infty$. By the lines above and (4.i) we have $l(C_n)=l(C)$, $l(C')\leqq\lim l(C_n)$ as $n\to\infty$, and thus $l(C')\leqq l(C)$. Analogously we can prove that $l(C)\leqq l(C')$, and thus $l(C)=l(C')$.

9. Fréchet distances of two curves. Given any two continuous curves in E_N, say, C: $x=f(u)$, $u\in I=[a, b]$, C_1: $x=g(v)$, $v\in J=[c, d]$, we shall consider the class $\{h\}$ of all homeomorphisms $u=h(v)$ from J onto I. Then for every $h\in\{h\}$ the expression $|f[h(v)]-g(v)|$, $v\in J$, has an absolute maximum $m[h]\geqq 0$ and we shall consider the infimum of this maximum $m[h]$ for all $h\in\{h\}$, say

$$\|C, C_1\| = \operatorname*{Inf}_h \max_{v\in J} |f[h(v)] - g(v)| .$$

We shall denote $\|C, C_1\|$ as the Fréchet distance of C to C_1. In other words,

$\|C, C_1\|$ is the infimum of all numbers $\epsilon \geqq 0$ having the following property: there exists a homeomorphism $u = h_\epsilon(v)$ from J onto I such that $|f[h_\epsilon(v)] - g(v)| \leqq \epsilon$ for all $v \in J$.

(9.i) *If C, C_1, C_2 denote continuous curves in E_N, we have* 1. $\|C, C_1\| \geqq 0$; 2. $\|C, C_1\| = \|C_1, C\|$; 3. $\|C, C_1\| \leqq \|C, C_2\| + \|C_2, C_1\|$; 4. $\|C, C_1\| = 0$ *if and only if* $C \sim C_1$.

The proof does not present difficulties. It should be pointed out here the rather trivial fact that the Fréchet distance $\|C, C_1\|$ of two curves C, C_1 is a quite different concept than the distance $\{[C], [C_1]\}$ of the two sets (graphs) $[C]$, $[C_1]$, the latter being defined, as usual, as the infimum of all distances $|x - y|$ between all pairs of points $x \in [C]$, $y \in [C_1]$. The inequality $\{[C], [C']\} \leqq \|C, C_1\|$ holds as an immediate consequence of the definitions. Obviously the sign $<$ may well hold. For instance, if $N = 2$, C: $x_1 = u$, $x_2 = 0$, $0 \leqq u \leqq 1$, C_1: $x_1 = 0$, $x_2 = 1 + u$, $0 \leqq u \leqq 1$, we have $\{[C], [C_1]\} = 1$, $\|C, C_1\| = \sqrt{5}$. For the two curves F_1, F_2 of Section 1 we have $[F_1] = [F_2]$, $\{[F_1], [F_2]\} = 0$, $\|F_1, F_2\| = 2$.

We may state now a remarkable extension of the property of lower semi-continuity of the Jordan length:

(9.ii) *If C, C_n, $n = 1, 2, \cdots$, are continuous curves in E_N, if $\|C_n, C\| \to 0$ as $n \to \infty$, then $l(C) \leqq \liminf l(C_n)$ as $n \to \infty$.*

Proof. Let C: $x = x(u)$, $u \in I$, C_n: $x = x_n(v)$, $v \in J_n$, $n = 1, 2, \cdots$, and $\delta_n = \|C, C_n\|$. Hence $\delta_n \to 0$ as $n \to \infty$. For every n there is a homeomorphism $v = h_n(u)$ such that $|x_n[h_n(u)] - x(u)| < \delta_n + n^{-1}$ for all $u \in I$, $n = 1, 2, \cdots$. Hence, if C_n': $x = X_n(u) = x_n[h_n(u)]$, $u \in I$, $n = 1, 2, \cdots$, we have $|X_n(u) - x(u)| < \delta_n + n^{-1}$ for all $u \in I$ and hence $X_n(u) \to x(u)$ as $n \to \infty$ for all $u \in I$. By (4.i) and (8.i) we have $l(C_n') = l(C_n)$, $l(C) \leqq \liminf l(C_n')$, and hence $l(C) \leqq \liminf l(C_n)$ as $n \to \infty$. Thereby (9.ii) is proved.

Curves C, C_1 (both oriented and open), which are F-equivalent, are often considered as "representations" of the same entity. To obtain this we shall denote by a *Fréchet curve* $\overline{C}$ (or F-curve) any family $\overline{C} = \{C\}$ of all curves C which are F-equivalent to one another. The equality of two F-curves, say $\overline{C} = \{C\}$, $\overline{C}_1 = \{C_1\}$, is denoted by $\overline{C} = \overline{C}_1$, and is simply defined by the identity of the two families $\{C\}$ and $\{C_1\}$. By F-distance $\|\overline{C}, \overline{C}_1\|$ of two F-curves $\overline{C} = \{C\}$, $\overline{C}_1 = \{C_1\}$, we shall denote the *F-distance* $\|C, C_1\|$ of any two curves $C \in \{C\}$, $C_1 \in \{C_1\}$. Such a number does not depend upon the particular choice of C and C_1 in the two families (because of (8.i)). Now theorem (9.i) has the following formulation:

If $\overline{C}$, $\overline{C}_1$, $\overline{C}_2$ denote F-curves we have (i) $\|\overline{C}, \overline{C}_1\| \geqq 0$ *and* $\|\overline{C}, \overline{C}_1\| = 0$ *if and only if* $\overline{C} = \overline{C}_1$; (ii) $\|\overline{C}, \overline{C}_1\| = \|\overline{C}_1, \overline{C}\|$; (iii) $\|\overline{C}, \overline{C}_1\| \leqq \|\overline{C}, \overline{C}_2\| + \|\overline{C}_2, \overline{C}_1\|$.

Thus all F-curves $\overline{C}$ form a metric space whose metric is the F-distance.

Also, if we denote by Jordan length $l(\overline{C})$ of an F-curve $\overline{C} = \{C\}$ the Jordan

length $l(C)$ of any one of the elements $C \in \{C\}$, then $l(\bar{C})$ does not depend upon the choice of C in $\{C\}$ (by (8.i)), and (9.ii) has the following formulation:

(9.iii) *If* $\bar{C}$, $\bar{C}_n$, $n=1, 2, \cdots$, *are F-curves, and* $\|\bar{C}_n, \bar{C}\| \to 0$ *as* $n \to \infty$, *then* $l(\bar{C}) \leqq \liminf l(\bar{C}_n)$ *as* $n \to \infty$.

10. The representation theorem.

(10.i) *For every continuous curve* C: $x=x(u)$, $a \leqq u \leqq b$, *with* $L=l(\mathfrak{C})<+\infty$, *there is another curve* C': $x=X(v)$, $a_1 \leqq v \leqq b_1$, *for which* $C' \sim C$, $X(v)$ *is an AC vector function, even Lipschitzian with constant* 1, *and*

$$l(C) = l(C') = \int_{a_1}^{b_1} | X'(v) | \, dv.$$

In other words: For every F-curve $\bar{C} = \{C\}$, with $L=l(\bar{C})<+\infty$, there is an element for which the Jordan length is given by the integral (2.2). In somewhat imprecise terms, we may say that each rectifiable curve has at least one representation for which the Jordan length is given by the integral (2.2).

Proof of (10.i). If $L=0$, then $x(u)=x_0$ is a constant, and we may assume $X(v)=x_0$ for all $a_1 \leqq v \leqq b_1$. Suppose $L>0$, and observe that $s(a)=0$, $s(b)=L$, and $s(u)$, $a \leqq u \leqq b$, is monotone nondecreasing and continuous in $[a, b]$. Also,

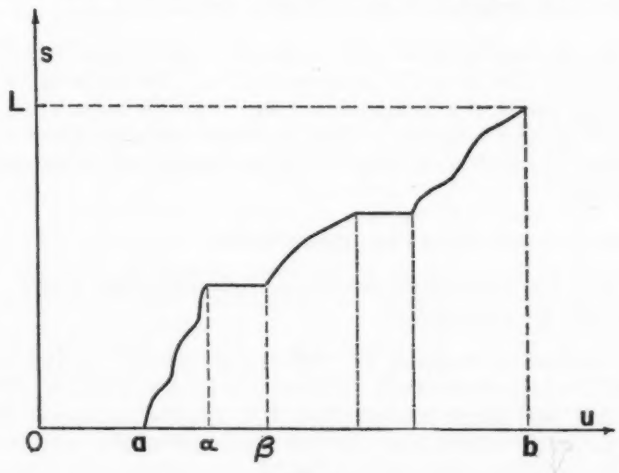

FIG. 4.

$s(u)$ is constant on an interval $[\alpha, \beta]$ if and only if $x(u)$ is constant there (Fig. 4). For each s_0, $0 \leqq s_0 \leqq L$, there is either a single point $u_0=\alpha=\beta$, or a maximal interval $\alpha \leqq u \leqq \beta$, $a \leqq \alpha \leqq \beta \leqq b$, with $s(u)=s_0$, and, in the latter case, $x(u)$ is constant on $[\alpha, \beta]$. Let $X(s_0)=x(u)$, $\alpha \leqq u \leqq \beta$. Then the vector function $X(s)$ is defined for

all $0 \leqq s \leqq L$. Let us prove that $X(s_0-0)=X(s_0)$ for every $0<s_0 \leqq L$. Indeed we have then $\alpha>a$ and, given $\epsilon>0$ there exists σ such that $0<\sigma \leqq \alpha-a$, and $|x(u)-x(\alpha)|<\epsilon$ for all $\alpha-\sigma \leqq u \leqq \alpha$. Since $[\alpha, \beta]$ is a maximal interval of constancy for $s(u)$, we have $s(\alpha-\sigma)<s_0$, *i.e.*, $\delta=s_0-s(\alpha-\sigma)>0$. For every s with $s_0-\delta \leqq s<s_0$, there is now some u with $\alpha-\sigma \leqq u \leqq \alpha$ and $X(s)=x(u)$. Thus $|X(s)-X(s_0)|=|x(u)-x(\alpha)|<\epsilon$, and hence $X(s_0-0)=X(s_0)$. Analogously we can prove that $X(s_0+0)=X(s_0)$ for every $0 \leqq s_0<L$. Thus $X(s)$ is continuous and $C': x=X(s), 0 \leqq s \leqq L$, is a continuous parametric curve. Let us prove that $C' \sim C$. Indeed, we may well approximate $s=s(u)$ by means of quasilinear, strictly increasing functions $s=\phi_n(u)$, $a \leqq u \leqq b$, $\phi_n(a)=0$, $\phi_n(b)=L$, with $|\phi_n(u)-s(u)|<1/n$ for all $a \leqq u \leqq b$, $n=1, 2, \cdots$. Then $X[\phi_n(u)] \to X[s(u)]$ uniformly in $[a, b]$ as $n \to \infty$, *i.e.*, for every given $\epsilon>0$ there is some n with $|X[\phi_n(u)]-X[s(u)]|<\epsilon$, or $|X[\phi_n(u)]-x(u)|<\epsilon$ for all $a \leqq u \leqq b$. Thus $C' \sim C$. We have now $|X(s')-X(s)|=s'-s$, hence, if $X=(X_1, X_1 \cdots, X_N)$, also $|X_r(s')-X_r(s)| \leqq s'-s$, for all $0 \leqq s<s' \leqq L$. Thus $X(s)$ is an AC vector function and also Lipschitzian with constant 1. By (8.i) and (6.i) we have $L=l(C)=l(C')=\int_0^L |X'(s)|\,ds$, where $|X'(s)| \leqq 1$. Since $0=\int_0^L [1-|X'(s)|]\,ds$, we conclude that $|X'(s)|=1$ almost everywhere in $[0, L]$.

Remark. The last line of the proof above shows that for every rectifiable curve C and a.e. in $[0, L]$, $L=l(C)$, we have $(dX_1/ds)^2+\cdots+(dX_N/ds)^2=1$. Since $ds/ds=1$, this relation is also a consequence of (6.ii).

A continuous mapping $C: x=p(u)$, $a \leqq u \leqq b$, is said to be *light* if for every $x \in [C]$, the set $p^{-1}(x)$ is totally disconnected, *i.e.*, its components are single points. In other words, $p(u)$ is light if and only if $p(u)$ is constant on no proper subinterval of $[a, b]$. Obviously $x=X(s)$ is a light mapping. Thus (10.i) yields that every rectifiable F-curve has a light representation. This statement is actually general:

(10.ii) *Every F-curve has a light representation.*

(10.iii) *Any two F-equivalent continuous light mappings* $x=f(u)$, $a \leqq u \leqq b$, $x=g(v)$, $c \leqq v \leqq d$, *are L-equivalent.*

For any continuous mapping $C: x=f(u)$, $a \leqq u \leqq b$, let $\Gamma=\{\gamma\}$ denote the collection of all maximal intervals of constancy for $f(u)$ in $[a, b]$ (*i.e.*, proper closed intervals, and single points). Then Γ is a decomposition of $[a, b]$ into disjoint parts, proper closed intervals and single points. For a light mapping, Γ is the collection of the single points of $[a, b]$. We shall think of Γ as ordered on $[a, b]$ in the natural order.

(10.iv) *If* $C_1: x=f(u)$, $a \leqq u \leqq b$, $C_2: x=g(v)$, $c \leqq v \leqq d$, *are continuous mappings, and* Γ_1, Γ_2 *are the corresponding collections, then* $C_1 \sim C_2$ *if and only if there exists a one-one ordered correspondence between* Γ_1 *and* Γ_2 *such that* $f(\gamma_1)=g(\gamma_2)$ *for every pair* $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ *of corresponding elements.*

11. The line integral as a Weierstrass integral. For curves $C: x=x(u)$, $a\leqq u\leqq b$, $x=(x_1, \cdots, x_N)$, for which the real functions $x_r(u)$, $r=1, \cdots, N$, are continuous, with their first derivatives $x_1', \cdots, x_N'$, and $x_1'^2+\cdots+x_N'^2>0$ everywhere on $[a, b]$, by line integral is often assumed, by definition, the integral

$$I(C,f)=\int_C f(x, t)=\int_a^b f[x(u), x'(u)]du, \tag{11.1}$$

where $f(x, t)$, $x=(x_1, \cdots, x_N)$, $t=(t_1, \cdots, t_N)$, is any given function of (x, t), continuous in (x, t) for all t and all $x\in[C]$, and thus I is said to be the line integral of f on C. In order to assure that the integral $I(C, f)$ has the same value on equivalent curves, *i.e.*, on different representations of the same Lebesgue, or Fréchet curve, the further condition is required (h): $f(x, Kt)=Kf(x, t)$ for all $K\geqq 0$, t, and $x\in[C]$ (this implies $f(x, 0)=0$ for all $x\in[C]$). Condition (h), *i.e.*, f positive homogeneous of degree one in t, is known to be sufficient and, essentially, necessary for the required invariance of I with respect to Lebesgue and Fréchet equivalence for all curves C. If $f(x, t)=|t|=(t_1^2+\cdots+t_N^2)^{1/2}$, then $I(C, f)$ is the length integral (Sec. 2, (2.2)); if $f(x, t)=f(x)t_1$, then $I(C, f)$ is the "work" of a force $f(x)$ in the direction of the x_1-axis; if $N=2$, C is closed, and $2f(x, t)=x_1t_2-x_2t_1$, then $I(C, f)$ is the "signed area" linked by the plane curve C.

It may be pointed out that $x(u)$ denotes the current point on C, that $x'(u)/|x'(u)|=[x_r'(u)/|x'(u)|, r=1, \cdots, N]$ denotes the vector of the direction cosines of the tangent t to C at $x(u)$, and that (11.1), by condition (h), can be written in the equivalent form $I(C, f)=\int_a^b f[x(u), x'(u)/|x'(u)|]|x'(u)|du$. Thus $I(C, f)$ depends on the chain of values taken by f in correspondence to each point $x(u)$ of C and the associate direction of the tangent t to C at $x(u)$.

Unfortunately, the definition of line integral by means of formula (11.1) is subject to the same criticism which was expressed in Section 2 for the length integral. Namely, definition (11.1), though adequate under the hypotheses of continuity and differentiability mentioned at the beginning of this section, is not adequate under weaker assumptions, for instance in the class of all (parametric, continuous) rectifiable curves C. In this general situation an integral (W-integral) can be defined which is quite adequate, and which coincides with the integral (11.1) whenever all functions $x_r(u)$, $r=1, \cdots, N$, are absolutely continuous (AC) in $[a, b]$ (thus certainly under the more restrictive assumptions mentioned at the beginning of the section). The W-integral, or Weierstrass integral, is defined by a process of limit similar to the one we have discussed for Jordan length (2.ii).

Let $C: x=x(u)$, $a\leqq u\leqq b$, $x(u)=[x_1(u), \cdots, x_N(u)]$, be any (parametric, continuous) rectifiable curve, let $D=[a=u_0<u_1<\cdots<u_n=b]$ be any subdivision of $[a, b]$ and δ_D the norm of D (Section 2). For each interval $[u_{i-1}, u_i]$ we may consider the two points $p_{i-1}=x(u_{i-1})=(x_{i-1,r}, r=1, \cdots, N)$ $p_i=x(u_i)=(x_{ir}, r=1, \cdots, N)$ on C, and, if $p_{i-1}\neq p_i$, the chord $s=p_{i-1}p_i$, whose

direction cosines are the N numbers $\alpha_{ir}=(x_{ir}-x_{i-1,r})$ solidus $|p_i-p_{i-1}|$. If α_i is the vector $\alpha_i=(\alpha_{i1}, \cdots, \alpha_{iN})$, then α_i may be thought of as the vector of the "average" direction cosines for the "arc $(p_{i-1}p_i)$ of C." If t_i denotes any point $u_{i-1}\leqq t_i\leqq u_i$, and $\bar{p}_i=x(t_i)$ is its image on the "arc $(p_{i-1}p_i)$ of C," then the sum

$$S_D = \sum_{i=1}^{n} f(\bar{p}_i, p_i - p_{i-1}) = \sum_{i=1}^{n} f(\bar{p}_i, \alpha_i)\,|p_i - p_{i-1}|$$

can be thought of as an "approximate" value for a line integral of f on C. Note that the intervals $[u_{i-1}, u_i]$ with $p_{i-1}=p_i$, *i.e.*, $p_i-p_{i-1}=0, f(\bar{p}_i, 0)=0$, have no bearing on the first of the sums above, and the same can be said for the second sum if we assume that the corresponding vector α_i is any arbitrary unit vector.

The following theorems hold

(11.i) *If $C: x=x(u)$, $a\leqq u\leqq b$, $x(u)=[x_r(u), r=1, \cdots, N]$, is any (parametric, continuous) rectifiable curve in E_N, if $f(x, t)$ is continuous in (x, t) for all t and $x\in[C]$, if $f(x, Kt)=Kf(x, t)$ for all $K\geqq 0$, t, $x\in[C]$, then the limit exists and is finite*

$$J(C, f) = \lim_{\delta\to 0} S_D = \lim_{\delta\to 0} \sum_{i=1}^{n} f(\bar{p}_i, p_i - p_{i-1}).$$

The limit $J(C, f)$ is said to be the Weierstrass integral of f on C, and it is proved that $C\sim C'$ implies $J(C, f)=J(C', f)$.

(11.ii) *Under the conditions above, if all functions $x_r(u)$, $r=1, \cdots, N$, are AC on $[a, b]$, then*

$$J(C, f) = \int_a^b f[x(u), x'(u)]du.$$

The last integral is said to be a Lebesgue-Tonelli integral. Because of (6.i) we can say that the Weierstrass integral $J(C, f)$ is given by the formula (11.1) (as an L-integral) whenever the Jordan length $l(C)$ of C is given by the length integral. By (10.i) every rectifiable Fréchet curve has a representation for which the Weierstrass integral is given by the Lebesgue-Tonelli integral.

Both theorems (11.i), (11.ii), and other properties of $J(C, f)$ are best proved as consequences of general theorems on interval functions, and the proofs are omitted. New proofs have been given by the author [L. Cesari, *Additive set functions and Weierstrass integral.*] The W-integral $J(C, f)$ was first proposed by Weierstrass, and then studied consistently by Tonelli as the main tool for the calculus of variations [L. Tonelli, *Rend. Acc. Lincei*, 21/1, 1912, 448–453, 554–559; 21/2, 1912, 132–137, *etc.*]. It was successively discussed by K. Menger, N. Aronszajn, G. Bouligand, and others. On the general topic of the present article the reader may consult L. Cesari, *Surface area*, quoted above, and T. Rado, *Length and area*, Amer. Math. Coll. Publ., vol. 30, 1948.

LOGIC AND MATHEMATICS COURSES

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The increasing tendency to introduce some work in logic into mathematics courses at the introductory level appears to me to need careful attention in terms of the aims and objectives of such introductory courses.

One of the first things that needs to be underlined is that for a great number of our undergraduates the introductory course is also the terminal course. This gives it a unique importance not shared by other courses. If the student is to learn anything about mathematics at all this is his only opportunity, for it is all too evident quite often that the mathematics he has had in the secondary school left him with no idea about mathematics except that of mechanical memorization and strange manipulations that somehow give the right answers. The introductory course for these students is the only place where they can begin to understand mathematics and its place in our civilization. Since I am writing this paper chiefly for mathematicians, I do not need to argue that no person can consider himself educated who does not to some degree comprehend mathematics and its role in creating, fashioning, and comprehending the kind of world we have.

Granted this need to give these students an understanding of what it is to do mathematics and what it is that mathematics does, the very utilitarian character of mathematics means that the student's education in other fields is impossible unless he acquires certain fundamental mathematical techniques. This too is so obvious that I shall not take the time to document it. What it means, however, is that these technical operations at least must be taught in the introductory course. For those who like myself prefer nonutilitarian reasons, I would suggest that an understanding of mathematics without some immersion in techniques is like trying to understand a youth in love by one who has never loved. It is, I know, the fashion today to decry what is called in derogatory tones "drill" or "technique," and to emphasize "thinking things out." This, as I shall show later, involves a basic misconception concerning the meanings of "thinking things out" and "technique." Be that as it may, one eliminates formulas and techniques only at the risk of doing students a disservice since it will make their education more difficult. Not even mathematics is so isolated and autonomous that it can be taught with neglect of its ramification. Even the "queen" must make her decisions in the light of the needs of her subjects.

It would be splendid if we could talk about our subject only in terms of this last group of students that I want to mention. These are those students for whom the introductory course lays the basis for a major and perhaps a career in mathematical research. For this group, quite obviously, the introductory course is not at all terminal but initiatory. Yet I venture to urge that the needs of this group are not exclusive but inclusive of those of the other two. These students need not only to understand the doing of mathematics and its applications, but they need to create in themselves the knack of technical skills and the fund of

knowledge without which they either become technical hacks or remain sterile. The basic difference between this group and the others is that this one needs more subject-matter and this perhaps cannot all be crowded into one course. Those who prepare for advanced work can, in any case, obtain the understanding and applications over a longer period of time. In a sense, however, I think the whole division I have made is quite artificial; but I will leave the problem of working out the details of a course for all three sets of students to others. The chief point I need to make is that there are these three aims of beginning courses in mathematics.

I have made this approach to my problem in order to have some basis for discussing the growing introduction of logic into introductory courses. Evidently, if one adds one or more chapters to an introduction to mathematics in which basic logical concepts are presented, there must be some justification other than enlarging the volume. It should be apparent, indeed, that to justify the inclusion of these notions one is best advised to endeavor to indicate that doing so will help achieve some desirable goal. Having stated a set of aims, we can now consider whether or not logical notions do make their attainment possible.

To help us in deciding our problem, let us look, rather sketchily to be sure, at the relation between logic and mathematics as it developed over the centuries. Logic, traditionally, was the science of correct reasoning. Mathematics was the science of quantity. The two, once upon a time, were separate studies. For many centuries the core of logic was the study of immediate inference and the syllogism. In later years Euler's and Venn diagrams were introduced and the interpretation of categorical propositions in terms of class-inclusion relations was added. So, throughout the passing centuries, logic was studied by those who wanted to learn how to derive valid conclusions from a given set of premises. Mathematics, on the other hand, was concerned to discover the properties of two kinds of quantities—the discrete and the continuous. In short, one who wanted to know the properties of space and time (which were quantities) learned mathematics. True, to derive the theorems of mathematics one did in fact use reasoning, still this was not so surprising. This attitude is quite prevalent. Let me quote one fairly-recent textbook in general college mathematics. "Mathematics is impossible without logic, for we must use correct arguments if we are to solve problems, prove theorems, *etc.*"

However, this quotation and the chapter on logic come at the end of the book. Indeed, if the student needs to be told that mathematics is impossible without logic in this sense, he must be quite stupid. Such a justification for the introduction of a chapter on logic which demonstrates no clear relation between logic and mathematics is as valueless as a chapter on logic in a textbook on surgery, for the justification would be exactly the same. In fact, this particular chapter to which I refer seems to me to be completely useless since it neither aids the student to understand mathematics nor gives any clear impression of the value of logic. It has no connection with the rest of the book, and I understand it is so considered by instructors who use it.

More significant reasons for including work in formal logic (or mathematical logic) can be given. In point of fact the relation between logic and mathematics is quite intricate, so much so that a true understanding of the nature of mathematics is virtually impossible without some ideas about logic. Even more significant is the fact that the distinction between logic and mathematics loses its sharpness, so that at times it is not possible to say whether one is working in logic or in mathematics. All of this is important even if we do not mention the application of symbolic logic to problems in algebras, statistics and computer-construction.

Credit is usually given to G. Frege for the first explicit and detailed attempt to derive the concept of natural number from logical notions. Frege actually was inspired by two objectives. He wished to construct logic in such a way that all the consequences of a set of statements could be derived purely mechanically. Deductive logic, he felt, needed to be just as much a calculus as arithmetic so that one could, by defining certain operations, perform all "reasoning" by manipulating the symbols according to these rules. Secondly, Frege sought to show that the number concept was at bottom constructible by definition from the elementary logical notions. He approached the problem by considering quantity, and in particular number, to be a property of groups (or classes of things) and a purely formal property at that. Since a class could be defined in terms of "propositional functions" by considering it to be composed of all those elements for which the function resulted in a true proposition, this rather obviously made it possible to start with logical notions and build up classes. Adding to this the notion of (1, 1) correspondence, Frege and later Russell came up with the definition of a number as "the class of all classes that can be put into (1, 1) correspondence with each other." It is plain to see at this point that this rather startling development, which was made better known by Russell in his *Philosophy of Mathematics* (Vol. I) and then with A. N. Whitehead in *Principia Mathematica* using a much better symbolism than had Frege, signifies that whoever wants to work at the roots of arithmetic needs to go back to logic. This means more than that. If the work of Frege and Russell was correct, then mathematics was but a branch of logic, and it is also possible to conceive of logic as mathematics. If logic could serve as a matrix for mathematical notions, could one not start with mathematics and define logical concepts in its terms? We will not try to answer this question but merely to point out that this paves the way for the expanded notions of "algebras," and the conception of an algebra as a logical system or a logical system as an algebra.

In a context such as this, I must mention George Boole. Boole too was inspired by the Leibnizean and Cartesian notion of a "universal science of order and measurement." He wanted to reduce the so-called "laws of reason" to mechanical operations. Accordingly, in his *Law of Thought*, Boole tried to do so and gave us what has been called "Boolean Algebra." This I mention simply to indicate again the close connection between algebra and logic and the stages that make it easy to identify the two.

The Frege-Russell derivation of finite arithmetic from the concept of class

(or function) was fortunately based upon a notion that led to the paradoxes. Russell formulated one of the earliest of these in the Russell paradox based on the definition of a particular class. Soon other paradoxes were published. The effect was to release a host of studies aimed at "saving" the foundations of mathematics. We need not describe these here except to indicate some of the things that resulted.

Attempts to correct the definition of a class to obtain one that would not lead to paradoxes brought about an intensive development of the theory of classes, both finite and transfinite. The notion of a class could be constructed or simply taken as an interpretation of a set of axioms. This in its turn complemented the development of axiomatic theory. The work of David Hilbert and his associates in this area is now part both of the history of logic and of mathematics. Out of it in large measure has come metamathematics and a deepening of our insight into what mathematics is. The result has been elaboration of such notions of "proof," "function," "rules of substitution," "rules of replacement," "rules of operation," analysis of recursion formulas, the distinction between variable and constant, and so on. Metamathematics is, in brief, the theory of mathematics and talks *about* mathematics. But above all it has tended to underline the notion of mathematics as the mechanical development of the deductive consequences of an initial set of statements. This explained and explains why the axiom-system is the ideal type of mathematical structure.

Hilbert's work on the axiomatization of geometry clarified the basic elements of such systems, too. It brought to the fore the importance of the problems of consistency, independence, completeness, and the nature of mathematical existence. The first of these are logical problems being concerned with the relation of implication. The last of these problems became of prime importance in the controversy with L. E. J. Brouwer, the Dutch mathematician, and centered around the principle of classical (or two-valued) logic known as the "law of excluded middle." It resulted in a clarification of such expressions in mathematics as "there exists a function such that . . . ." But it also led into the whole theory of logical matrices by means of which the multivalued logics were developed, and which play an important role in the construction of certain types of computers.

All of these developments brought out the fact that pure mathematics ultimately reduces to the mechanical (*i.e.*, deductive) manipulation of sets of symbolic arrangements according to pre-established rules. This is why I said earlier in this paper that there is a fundamental misconception involved in the attack on drill and technique. If pure mathematics teaches one to think, it instructs in the art of mechanical elaboration of initial premises. Facility in such work, as in all mechanical operations, comes through repetition. To get the "feel" of such manipulation, even when one is simply trying to understand what mathematics is rather than to become an expert mathematician, one does need to do this sort of work. But I do not want to involve myself in debating

this issue. It is, I believe, to the realm of applied mathematics that those refer who speak in terms of "thinking things through." In this area we enter upon the significance of mathematics for our civilization. This is, however, a different matter.

All of these remarks make it quite evident that the interrelations between symbolic logic and mathematics are many and intricate. It is, in my opinion, simply impossible to understand the nature of mathematics without some introduction to the various calculi of symbolic logic. It is more and more evident that both from a practical as well as a theoretical view, symbolic logic is of importance to mathematicians. This calls for a fundamental revision of introductory courses in the direction of an introduction of the basic concepts and procedures of symbolic logic. To make room for this, something clearly will need to be omitted from the standard general mathematics course. The two alternatives have been either to cut out so-called more difficult and useless material or to cut down on the drill. Both of these are mistaken directions. Rather, if one examines standard-type texts and sees the vast amount of work devoted to so-called "review of fundamentals," the possibility arises of doing something here. There are always three or four chapters that include material that students should have had in high school. This could easily be replaced by other material. I believe it is not too much to ask of college freshmen, especially in these days of mass influx and limited facilities, to be able to perform the fundamental operations, to work with linear equations and to graph elementary functions. But one must warn against simply replacing chapters devoted to these reviews with some devoted to logic with no organic relation between the logic and the mathematics indicated. An excellent example of the sort of thing that can be done is found in the recent book by John Kemeny, J. L. Snell and Gerald Thompson entitled *Introduction to Finite Mathematics.** As I read it, it meets the three needs I have indicated. It begins with logic and shows the relation of logical to mathematical notions, thereby making for a better understanding of the nature of mathematics. It devotes much time to probability, matrices, and sets, providing a basis for further work in mathematics, and it devotes attention to the application of mathematics in the behavioral sciences, giving insight into the place of mathematics in our civilization.

Knowledge of symbolic logic will inevitably profit both the professional mathematician, pure and applied, and the beginning student for it will help

* John G. Kemeny, J. Laurie Snell, Gerald L. Thompson, Introduction to Finite Mathematics, 1957.

Cf. also for the relation of logic to mathematics, E. R. Stabler, An Introduction to Mathematical Thought, 1953; and for an approach via logic, Patrick Suppes, Introduction to Logic, 1957.

For a general introduction to the foundations of mathematics cf. L. O. Kattsoff, A Philosophy of Mathematics, Iowa University Press, 1948.

For a rather technical study cf. S. C. Kleene, Introduction to Metamathematics, Princeton, 1952.

For a good introduction to set-theory, cf. A. Fraenkel, Abstract Set Theory, 1955.

display to them what McShane has called the "two legs" on which mathematics stands—its "innate beauty and austere elegance" as well as its "usefulness to scientists and technicians of all kinds."*

* E. J. McShane, Maintaining communication, this MONTHLY, vol. 64, 1957, pp. 309–317.

PSEUDO-INVERSES IN ASSOCIATIVE RINGS AND SEMIGROUPS

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1. Introduction. E. H. Moore ([5], and *cf.* also Penrose [6], Rado [8]) has shown how, given any square matrix x with elements in the complex field, it is possible to define another matrix, coinciding with x^{-1} whenever x^{-1} happens to exist, and having (even for singular x) properties leading to simple proofs of several theorems which yield to other methods only with difficulty (if at all). More precisely, Moore's result (stated in Penrose's form) is that, given any square* complex matrix x, then there is always exactly one complex matrix c satisfying

$$cxc = c \quad and \quad xcx = x,$$

$$cx \text{ and } xc \text{ are both hermitian.}$$

Following Penrose, we denote this unique matrix c by $x^\dagger$, and call it *the generalized inverse of* x; for evidence of the value of this concept, we refer the reader to the work of Moore and Penrose ([5], [6], [7]).

It is clear from Penrose's arguments that the function $x^\dagger$ thus defined for square complex matrices can be extended to the elements of any finite-dimensional algebra R over the complex field by the standard device of mapping R isomorphically into the algebra M_n of all $n \times n$ complex matrices (*e.g.* with $n = 1 + \dim R$), *provided* that the image of R in M_n admits the operation of forming transposed complex conjugate matrices. However, there seems to be no prospect of extending Moore's generalized inverse concept so as to apply to finite-dimensional algebras over a given division ring F unless F at any rate admits an involutory anti-automorphism $\lambda \to \bar{\lambda}$ such that $\lambda_1\bar{\lambda}_1 + \cdots + \lambda_m\bar{\lambda}_m = 0$ implies $\lambda_1 = \cdots = \lambda_m = 0$. (This, of course, would restrict F to having zero characteristic.)

In this note we show how to define a function, somewhat analogous to the generalized inverse function, over the elements of *arbitrary* finite-dimensional algebras, and even of an extensive class of associative rings. Indeed, all of our

* In fact, Moore and Penrose considered rectangular matrices, but this aspect of the result is not relevant here.

theorems in Section 2 below apply equally well to arbitrary semigroups; however, we shall for brevity, generally use the language of ring theory.

Given any associative ring R, and any element x of R, we shall call c *pseudo-invertible* (*in* R) if an element c of R exists satisfying

(i) $cx = xc$,

(ii) $x^m = x^{m+1}c$ *for some positive integer* m,

and

(iii) $c = c^2x$.

By Theorem 1 below, these three conditions determine c uniquely when it exists, so we may refer to c as *the* pseudo-inverse of x. We show also (*inter alia*) that the existence of such a c in fact follows from the apparently much weaker hypothesis that elements a, b of R exist such that

$$x^p = x^{p+1}a, \qquad x^q = bx^{q+1}$$

(for some positive integers p, q); in other words, pseudo-invertibility (as applied to a given element x of a given ring) coincides with Azumaya's property [2] of *strong* π*-regularity*. Since it is known that every element of any algebraic ring (as defined in [3]), or of any ring with minimal condition on left or right ideals, is strongly π-regular, it follows that our pseudo-inverse function is applicable to all such rings (and, in particular, to matrices and all finite-dimensional algebras). The specialization of pseudo-invertibility in which (ii) above holds with $m=1$ was discussed by Azumaya, and indeed some of our arguments below are, formally, only slight generalizations of his: the essence of our present contribution is that (i), (ii) and (iii) determine c uniquely even when m is unrestricted (and indeed variable). The case $m=1$ of pseudo-invertibility, but with (iii) omitted, was discussed by A. H. Clifford [10], who called the corresponding property "relative invertibility"; in view of our Theorem 4 below, this property of relative invertibility is, in fact, precisely equivalent to pseudo-invertibility with $m=1$.

In our third section, we use pseudo-inverses to obtain a simple proof of another result of Azumaya ([2] Theorem 4). At the suggestion of a referee, we describe, in a concluding section, another way of looking at pseudo-inverses in any semigroup S: it turns out that our definition can be equivalently expressed, in a natural and simple way, in terms of certain idempotents and maximal subgroups of S. After Moore's and Penrose's work, it seems not unreasonable to hope that the pseudo-inverse may prove to be a useful tool in dealing with those rings in which it is everywhere defined, and that the publication here of the basic properties of pseudo-inverses may perhaps serve as a starting-point for other applications.

2. General results on pseudo-inverses. In this section we establish a number of facts about pseudo-inverses in rings (or semigroups) not subjected to any global conditions. First and most fundamentally, we have

THEOREM 1. *Let R be any given associative ring (or semigroup), and x any element of R. Then x has at most one pseudo-inverse in R, and, if a pseudo-inverse for x does exist, it commutes with every element of R which commutes with x.*

Proof. Let c_1, c_2 satisfy conditions corresponding to those imposed on c by (i), (ii) and (iii) of our definition of pseudo-invertibility, say with integers m_1, m_2 in (ii). Then, on writing $m=\max(m_1, m_2)$, the conditions on c_1, c_2 arising from (i), (ii) certainly imply

$$c_1x^{m+1} = x^m = x^{m+1}c_2, \tag{A}$$

while those arising from (i), (iii) certainly imply

$$c_1 = \overset{2}{c_1}x, \qquad c_2 = x\overset{2}{c_2}. \tag{B}$$

And in fact (A), (B) together imply that $c_1=c_2$. For, by induction from (B), we have $c_1=c_1^{k+1}x^k$, $c_2=x^kc_2^{k+1}$ $(k=1, 2, \cdots)$, and in particular $c_1=c_1^{m+1}x^m$, $c_2=x^mc_2^{m+1}$, whence, by (A),

$$c_1 = \overset{m+1}{c_1}\overset{m}{x} = \overset{m+1}{c_1}\overset{m+1}{x}c_2 = c_1xc_2 = \cdots = c_2.$$

Thus x has at most one pseudo-inverse c. Also, if y is any element of R satisfying $xy=yx$, then, by (ii) and (i), we find

$$cx^my = cyx^m = cyx^{m+1}c = cx^{m+1}yc = x^myc,$$

whence $c^{m+1}x^my=x^myc^{m+1}$; but, as above, (iii) gives $c=c^{m+1}x^m$, and so, using (i) again, we conclude that

$$cy = c^{m+1}x^my = x^myc^{m+1} = yc^{m+1}x^m = yc,$$

as required.

In view of Theorem 1, we may denote the unique pseudo-inverse of a given pseudo-invertible element x by x'; obviously, whenever x^{-1} exists in the ordinary sense, then x' exists and $x'=x^{-1}$. It should be noted that, when it is defined at all, x' is independent of what ring we think of x as lying in (in the sense that, if R_1, R_2 are subrings of a given ring R and x lies in $R_1\cap R_2$, having pseudo-inverses c_1, c_2 in R_1, R_2 respectively, then c_1, c_2, being pseudo-inverses for x in R, must coincide).

COROLLARY 1. *If $x_1, \cdots, x_j$ are given pseudo-invertible elements (of some ring) with $x_sx_t=0$ $(s, t=1, \cdots, j;\ s\neq t)$, then $x_1+\cdots+x_j$ is also pseudo-invertible, with $(x_1+\cdots+x_j)'=x_1'+\cdots+x_j'$.*

Proof. There will clearly be no loss of generality in supposing from the outset that $j=2$; and it will also be convenient to write u, v in place of x_1, x_2. By (i), the hypothesis $uv=vu$ $(=0)$, and a double application of the last part of Theorem 1, we see that u, v, u', v' all commute with one another; further, by (iii),

any product of these which simultaneously involves u or u' and also v or v' can be expressed as a product involving both u and v, and must consequently vanish. Hence

$$(u' + v')(u + v) = (u + v)(u' + v'),$$

$$(u' + v')^2(u + v) = u'^2u + v'^2v = u + v,$$

and, choosing m so large that $u^m = u^{m+1}u'$, $v^m = v^{m+1}v'$, we have

$$(u + v)^{m+1}(u' + v') = u^{m+1}u' + v^{m+1}v' = u^m + v^m = (u + v)^m;$$

by the first part of Theorem 1, the result now follows.

Given any pseudo-invertible element x of a ring, then, by (ii), there will be a unique (positive) integer $i(x)$ such that $x^m = x^{m+1}x'$ for every $m \geqq i(x)$ but for no $m < i(x)$. We shall refer to $i(x)$ as *the index of* x; again, provided that it does indeed exist, this integer $i(x)$ does not depend on what ring we regard x as lying in. If x is not pseudo-invertible, then we take $i(x) = \infty$ conventionally.

The special case of Theorem 1 in which $m_1 = m_2 = 1$ is, apart from differences in terminology, just Azumaya's ([2] Lemma 1) with the existence clause omitted;* our next result, however, is of interest chiefly when $i(x) > 1$.

THEOREM 2. *Let x be any pseudo-invertible element (of some given ring) and k any positive integer. Then x^k is pseudo-invertible, with $(x^k)' = (x')^k$, and $i(x^k)$ is the unique (positive) integer q satisfying $0 \leqq kq - i(x) < k$.*

Proof. By (i), $x^k(x')^k = (x')^k x^k$. Also, by induction from (ii), (iii) respectively, we have $x^{i(x)} = x^{i(x)+j}(x')^j$, $x' = (x')^{j+1}x^j$ $(j = 1, 2, \cdots)$, so that, since $kq \geqq i(x)$,

$$(x^k)^q = x^{kq-i(x)}x^{i(x)} = x^{kq-i(x)}x^{i(x)+k}(x')^k = (x^k)^{q+1}(x')^k,$$

$$(x')^k = (x')^{k-1}(x')^{k+1}x^k = ((x')^k)^2x^k.$$

Thus $(x')^k$ satisfies the conditions for $(x^k)'$, and $i(x^k) \leqq q$.

Finally, $i(x^k) < q$ would mean that $(x^k)^{q-1} = (x^k)^q(x')^k$, and, since $x' = x^{k-1}(x')^k$, this in turn implies that $x^{k(q-1)} = x^{kq-(k-1)}x' = x^{k(q-1)+1}x'$, whence, by the definition of $i(x)$, we should have $k(q-1) \geqq i(x)$, contrary to our definition of q.

THEOREM 3. *Given any element x of a ring, then if x is pseudo-invertible so is x'; in fact x' has index 1 whenever it exists, and then $x'' = x^2x'$.*

To prove this, one has merely to verify that, if c satisfies (i), (ii) and (iii), then $d = x^2c$ satisfies $cd = dc$, $c = c^2d$, $d = d^2c$. We omit the details, and also leave to the reader the proofs of the following equally trivial joint corollaries of Theorems 1, 2 and 3:

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* The last clause of Theorem 1 in this special case has also been set as a problem by T. Skolem [9].

COROLLARY 2. *Given any element x of a ring R, then $x''=x$ if and only if x is pseudo-invertible with index 1; and, when this is the case, for any given element y of R, x commutes with y if and only if x' does.*

COROLLARY 3. *Given any pseudo-invertible element x of a ring, then $(x^k)''=x^k$ for every integer $k \geqq i(x)$.*

COROLLARY 4. *For any pseudo-invertible element of a ring, $x'''=x'$.*

We have already noted that an element x of a ring R is called *strongly π-regular* in R if elements a, b of R and positive integers p, q exist such that

$$x^p = x^{p+1}a, \qquad x^q = bx^{q+1}. \tag{1}$$

We now prove

THEOREM 4. *Given any element x of a ring R, then x is pseudo-invertible in R if and only if it is strongly π-regular in R.*

Proof. That pseudo-invertibility implies strong π-regularity is obvious (even without using (iii)). Conversely, we shall show that (1) implies (i), (ii) and (iii) with $m=\max(p, q)$ and $c=x^m a^{m+1}$.

We note first that (1) gives

$$x^{m+1}a = x^m = bx^{m+1},$$

so that $x^m a = bx^{m+1}\cdot a = b\cdot x^{m+1}a = bx^m$, whence, by induction, $x^m a^h = b^h x^m$ $(h=1, 2, \cdots)$. Thus our choice $c=x^m a^{m+1}$ can equivalently be written $c=b^{m+1}x^m$, and we have:

(i) $xc = x\cdot x^m a^{m+1} = x^{m+1}a\cdot a^m = x^m a^m = b^m x^m = \cdots = cx$ by symmetry;

(ii) By another induction, $x^m = x^{m+k}a^k$ $(k=1, 2, \cdots)$, and so

$$x^{m+1}c = x^{m+1}\cdot x^m a^{m+1} = x^{m+(m+1)}a^{m+1} = x^m;$$

(iii) By (i) and (ii) which we have just proved,

$$c^2 x = c\cdot xc = c\cdot x^{m+1}a^{m+1} = x^{m+1}c\cdot a^{m+1} = x^m a^{m+1} = c.$$

It is clear from this proof that (1) ensures that $i(x) \leqq \max(p, q)$, and (cf. [2] Lemma 3) indeed it is easy to see that in fact $i(x) \leqq \min(p, q)$: for if, say, $p<q$, then (1) gives $x^p = x^{p+(q-p)}a^{q-p} = x^q a^{q-p}$, so that $bx^{p+1} = bx\cdot x^q a^{q-p} = bx^{q+1}\cdot a^{q-p} = x^q a^{q-p} = x^p$. In view of this, our Theorem 4 includes Azumaya's Theorem 3, in which he proved that strong π-regularity implies the *existence* of c satisfying (i) and (ii); we can arrange for uniqueness only by introducing some additional restriction such as (iii). For a discussion of the implications subsisting between strong π-regularity (*i.e.*, pseudo-invertibility), right π-regularity, left π-regularity and π-regularity, we refer the reader to Azumaya's paper.

COROLLARY 5. *Let R be any finite-dimensional algebra. Then, for any given $x \in R$, x' exists and lies in the subalgebra generated by x.*

Proof. If R has dimension k, then $x, x^2, \cdots, x^{k+1}$ are linearly dependent, and so, for some $j \leqq k+1$, x^j is a linear combination of $x^{j+1}, x^{j+2}, \cdots$, hence even of x^{j+2} and higher powers. Thus x is strongly π-regular, with $a=b$ in (1) expressible as polynomials in x (without constant terms); the corollary now follows from the form of c in the proof of Theorem 4 above. More generally, the result clearly holds even for all "algebraic rings" (in which, by definition, there corresponds to each element x an integer $j(x)$ such that $x^{j(x)}$ is a linear combination of $x^{j(x)+1}$ and higher powers).*

We remark that, immediately from the definition and uniqueness property, the operation of taking the pseudo-inverse (when such exists) of a given element commutes with all homomorphisms and antihomomorphisms of the containing ring. In particular, for any matrix algebra over the complex field, the pseudo-inverse of the complex conjugate (or transpose) of a given matrix is the same as the complex conjugate (or transpose) of the pseudo-inverse. Thus the pseudo-inverse of a given complex matrix x is real (symmetric, hermitian, *etc.*) whenever x is; and, even for arbitrary square complex matrices, it can be shown (*e.g.*, by using (iii) and Corollary 5) that the property of having all eigenvalues real and nonnegative is also preserved.

3. Sufficient conditions for pseudo-invertibility in rings.

Following Azumaya, we call an element x of a ring R *right π-regular* in R if a positive integer p and an element a of R exist such that $x^p = x^{p+1}a$. In these circumstances we shall define the *right index of* x in R, denoted by $r(x)$ (or more precisely $r_R(x)$), as the smallest integer p occurring in any such representation; and we take $r(x) = \infty$ whenever x is not right π-regular. Thus $r(x)$ is defined for *all* elements of R; and we define the *left index* $l(x) = l_R(x)$ of x in R similarly. Then, for any given x, we know from Theorem 4 that $i(x)$ is finite (*i.e.*, x is pseudo-invertible) if and only if $r(x)$, $l(x)$ are both finite, while our remark following Theorem 4 shows that then in fact§ $r(x) = l(x) = i(x)$.

This "local" result (*i.e.*, concerning only the single element x) is new, but a still more striking "global" result has been known for some time ([4] p. 74). Given any subset T of a ring R, let us write

$$i(T) = \sup i(x), \qquad r(T) = \sup r(x), \qquad l(T) = \sup l(x),$$

where each upper bound is taken over all $x \in T$ (with the natural convention for infinite values‡). Obviously $r(T) \leqq i(T)$, and Kaplansky's result (or rather

* In the case of a square matrix x over an algebraically closed field F, we could alternatively prove the existence of x' by combining Corollary 1 (for $j=2$) with the well-known fact that x is similar over F to diagonal (u, v), where u, v are square matrices over F with u (if occurring) nilpotent and v (if occurring) nonsingular.

§ For a pseudo-invertible element x, the common value of $r(x)$, $l(x)$ and $i(x)$ coincides with the value of the "index of x" as defined by Azumaya (whose definition concerned only pseudo-invertible elements); but it is important in what follows that $r(x)$, $l(x)$ and $i(x)$ are meaningful for *arbitrary* x, and that $r(x)$, $l(x)$ (separately) can be finite even for nonpseudo-invertible x.

‡ More precisely, we take $i(T) = \infty$ whenever the $i(x)$ with $x \in T$ are finite but unbounded, and also whenever $i(x) = \infty$ for some $x \in T$; and similarly for $r(T)$, $l(T)$.

a slightly-strengthened form of it) is that, if $r(R)$ is finite, then so is $i(R)$ and in fact $i(R)=r(R)$; moreover, by our final remark in the previous paragraph, we even have $i(x)=r(x)=l(x)<\infty$ for each $x\in R$.

To motivate the next theorem (which was discovered by Azumaya, and does not seem readily extensible to semigroups), we note first that our definitions of $i(x)$, $r(x)$, $l(x)$ are of course consistent with the accepted meaning of the "index" of a given nilpotent element of a ring. More precisely, every nilpotent element x is clearly pseudo-invertible (with $x'=0$), so that $i(x)=r(x)=l(x)<\infty$, and satisfies $x^{i(x)}=0$, $x^{i(x)-1}\neq 0$ (since $x^{i(x)}=x^{i(x)+k}(x')^k$ for arbitrarily high k, while $x^{i(x)-1}=0$ would contradict the definition of $i(x)$). Thus, if $N=N(R)$ denotes the set of all nilpotent elements of the ring R, then we always have $i(N)=r(N)=l(N)$ (possibly all infinite), and $i(N)<\infty$ expresses the condition that the nilpotent elements of R have bounded index.

In ([2] Theorem 4), Azumaya generalized Kaplansky's result that the finiteness of $r(R)$ implies that of $i(R)$; replacing the finiteness of $r(R)$ by that of $i(N)$, he showed that this obviously weaker hypothesis in fact implies that $i(x)\leqq i(N)$ whenever $r(x)$ (or $l(x)$) is finite. We present now a new and very simple proof, bearing only a slight resemblance to Azumaya's (and none to Kaplansky's), of this result.

THEOREM 5. *Let R be any associative ring whose nilpotent elements have bounded index (i.e., with finite bound $i(N)$). Then every right π-regular element x of R is pseudo-invertible, with $i(x)=r(x)=l(x)\leqq i(N)$.*

Proof. Given any right π-regular element x of R, say with $x^p=x^{p+1}a$, we shall first show that x is necessarily pseudo-invertible; by Theorem 4, it will be enough to find $b\in R$ and a positive integer q such that $x^q=bx^{q+1}$.

Now, by induction, we have $x^p=x^{p+k}a^k$, and so

$$x^{p+k}(x^p-a^kx^{p+k})=(x^p-x^{p+k}a^k)x^{p+k}=0 \qquad (k=1, 2, \cdots).$$

Also each of the 2^t monomials in the expansion of $(x^p-a^kx^{p+k})^t$ has x^{p+k} as a right-hand factor provided only that $pt\geqq p+k$, and so, for each k, we have $(x^p-a^kx^{p+k})^{t+1}=0$ for large enough t. Hence, by our hypothesis that $i(N)$ is finite, it follows that $(x^p-a^kx^{p+k})^{i(N)}=0$, so that $x^{i(N)p}\in Rx^{p+k}$ $(k=1, 2, \cdots)$. Choosing $k=(i(N)-1)p+1$, we deduce that an element b of R exists such that $x^q=bx^{q+1}$, where $q=i(N)p$.

Thus x is pseudo-invertible, so that (i) and (ii) clearly give $(x-x^2x')^m=0$ for some m, whence $(x-x^2x')^{i(N)}=0$; hence, by (i), $x^{i(N)}=x^{i(N)+1}y$ for a suitable polynomial y in x, x' (with integer coefficients and no constant term), and so, finally, by (i) and the proof of Theorem 4, we can conclude that $i(x)\leqq i(N)$. (Also, incidentally, we have the explicit representation $x'=x^pa^{p+1}$).

COROLLARY 6. *Let R be any associative ring whose nilpotent elements have bounded index. Then, if every element of R is right π-regular, R must in fact be boundedly right π-regular, i.e. $r(R)$ is finite, and indeed $r(R)=i(R)=i(N)<\infty$.*

To deduce this from the theorem, we have only to note that $r(R) \leqq i(R) \leqq i(N) = r(N) \leqq r(R)$ (the inequality $i(R) \leqq i(N)$ following from Theorem 5 and the hypothesis of the corollary, while the other relations are obviously true in any ring); and of course Kaplansky's result is included in Corollary 6. We remark also that, by Azumaya's ([2] Theorem 5), one may, in the hypothesis of Corollary 6, replace "right π-regular" by "π-regular."

Some familiar examples of rings satisfying the hypotheses of Corollary 6 are (a) rings with minimal condition on (say) right ideals, and (b) algebraic rings of bounded degree (or, more generally, of bounded index); but it is already known (see [1] Theorem 3.1 and [4] p. 74) that rings of type (a) have $r(R)$, $l(R)$ finite, and this is obvious in case (b). However, Arens and Kaplansky's proof that $r(R)$ is finite in case (a), though brief enough, depends on the Wedderburn-Artin structure theory, and it would be gratifying to find a direct proof. We mention in this connection that, since every element x of any ring of type (a) is (from consideration of the descending chain of principal right ideals generated by the powers of x) clearly right π-regular, it would, in view of Corollary 6, suffice to prove (directly) the finiteness of $i(N)$; and, for this, since the rings in question have nilpotent (Jacobson) radicals, only the semisimple case need be considered.

4. Concluding remarks. The author is grateful to a referee for pointing out that, even in the context of arbitrary semigroups, the definition of pseudo-invertibility can be rephrased in a way that may perhaps make it more immediately available as a tool for tackling certain problems. To do this, given any element e of a semigroup S, let G_e denote the set of all elements $a \in S$ such that

$$(\alpha)\ ae = ea = a, \quad \text{and} \quad (\beta)\ ab = ba = e \text{ for some } b \in S.$$

Clearly, if G_e is nonempty, then e is necessarily idempotent, and, conversely, we have (this being, essentially, Lemma 1.3 of Clifford's paper [10]).

THEOREM 6. *If e is idempotent, then G_e is a group with e as identity element, and moreover every subgroup G of S containing e is in fact a subgroup of G_e.*

Proof. To prove the first assertion, it clearly suffices to show that, given any $a \in G_e$, we can find an inverse for a in G_e; and (e being idempotent) this is immediate, since (α), (β) for a and any corresponding b imply that $ebe \in G_e$, so that ebe is the inverse of a in G_e.

For the second assertion, we have only to note that e must be the identity of any subgroup G in which it lies, so that (α), (β) are immediate for all $a \in G$, as required.

COROLLARY 7. *The maximal subgroups of S are precisely the subgroups G_e with e idempotent.*

We next have

THEOREM 7. *Given any element x of a semigroup S and any positive integer m, then the statement*

(1) *x is a pseudo-invertible element of S with index at most m*

is equivalent to

(2) *there is an idempotent $e \in S$ such that $ex = xe \in G_e$ and $x^m \in G_e$.*

Proof. To see that (1) implies (2), we mention first that (i) and (iii) of the definition of pseudo-invertibility imply that $e = xc = cx$ is idempotent. Also, by (i), $ex = xe$ and $(ex)e = e(ex) = ex$, while (iii) gives $(ex)c = c(ex) = e$; hence $ex = xe \in G_e$. Similarly, (ii) becomes $x^m = x^m e$, so that $x^m e = ex^m = x^m$, while (i) and (iii) give $x^m c^m = c^m x^m = e$, whence $x^m \in G_e$ and (2) follows.

Conversely, given (2), let c denote the inverse of $ex = xe$ in the group G_e. Then $ce = ec = c$ and $cex = xec = e$, whence $cx = xc = e$ and $c = ce = c^2 x$, which are (i) and (iii). Finally, if $x^m \in G_e$, then we have $x^m = x^m e = x^{m+1} c$, which is (ii).

COROLLARY 8. *An element x of S is pseudo-invertible with $i(x) = 1$ if and only if x belongs to some maximal subgroup of S.*

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ACKNOWLEDGMENT

At the time of publication, the author of *Solution of a ranking problem from binary comparisons*, this MONTHLY, vol. 64, No. 8, Part II, October, 1957, was unaware of the work of R. A. Bradley and M. E. Terry along similar lines. Had he known of this, he would have made reference to their paper, *The rank analysis of incomplete block designs* I; *The method of paired comparisons*, Biometrika, vol. 39, Parts 3 and 4, December, 1952, both in the body of the text and in the bibliography.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the eighteenth William Lowell Putnam Mathematical Competition held on February 8, 1958, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of the Polytechnic Institute of Brooklyn, Brooklyn, New York. The members of the team were Martin Isaacs, Donald Passman, and Lawrence A. Shepp; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Jonathan L. Alperin, Everett C. Dade, and Stephen Lichtenbaum; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were David R. Brillinger, (Mrs.) Marion Green, and Carl R. Riehm; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of Manitoba, Winnipeg, Manitoba. The members of the team were Ian Connell, Wilbur Jacob Jonsson, and Philip James Peebles; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are David R. Brillinger, University of Toronto; Donald J. C. Bures, Queen's University; Richard Dudley, Harvard University; Joseph Lipman, University of Toronto; and Lawrence A. Shepp, the Polytechnic Institute of Brooklyn. Each of these will receive a prize of fifty dollars.

The six succeeding persons (tenth and eleventh tied) ranking highest in the examination, named in alphabetical order, are Robert Hartshorne, Harvard University; John R. F. Hewett, University of Toronto; Stanley Kaplan, Cornell University; H. Jerome Keisler, California Institute of Technology; Stephen Lichtenbaum, Harvard University; and J. G. Petersen, Stanford University. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being Luis Baez-Duarte, H. Jerome Keisler, and Charles J. Stone; College of the City of New York, New York, New York, the members of the team being Samuel Klein, Eugene Luks, and David Shelupsky; Rice Institute, Houston, Texas, the members of the team being David M. Dahm, Donald J. Deckard, and

Burton Randol; and Wesleyan University, Middletown, Connecticut, the members of the team being Benjamin Day, Justus Diller, and Robert T. Fisher.

Fourteen individuals were given honorable mention. The names, alphabetically arranged, are: Everett C. Dade, Harvard University; David M. Dahm, Rice Institute; Ronald Graham, University of Alaska; George Hannauer, Oberlin College; Paul B. Kantor, Columbia University; Samuel Klein, College of the City of New York; Alan Vernon Lemmon, Washington University; Dale Nelson, Knox College; Guillermo Owen, Fordham University; Donald Passman, Polytechnic Institute of Brooklyn; Philip James Peebles, University of Manitoba; Carl R. Riehm, University of Toronto; William Silvert, Brown University; Harold Nathaniel Ward, Swarthmore College.

A total of 539 individuals from 96 institutions entered the competition this year. Of this number 109 individuals and 3 institutions were unable to compete, due to various reasons. Therefore, a total of 430 undergraduates from 93 institutions actually took part in the competition.

The following is a list, in alphabetical order, of all colleges and universities which entered teams in the competition: Agricultural and Mechanical College of Texas, Alabama Polytechnic Institute, Arizona State College (Flagstaff), Arizona State College (Tucson), Blackburn College, Boston University, Brandeis University, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Case Institute of Technology, Central Michigan College, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Fairleigh-Dickinson University, Fordham University, Geneva College, Georgia Institute of Technology, Harvard University, Incarnate Word College, Kenyon College, Knox College, Lebanon Valley College, Livingstone College, Massachusetts Institute of Technology, McGill University, Mississippi State College, New York University, Oberlin College, Polytechnic Institute of Brooklyn, Princeton University, Purdue University, Queen's University (Kingston, Ontario), Radcliffe College, Sacramento State College, Saint Francis Xavier University, Saint Martin's College, Saint Olaf College, San Jose State College, Southwestern Louisiana Institute, Stanford University, State College of Washington, Stevens Institute of Technology, Swarthmore College, The Cardinal Stritch College, The College of Saint Thomas, The College of the City of New York, The Rice Institute, Tufts University, University of Buffalo, University of British Columbia, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Detroit, University of Houston, University of Illinois, University of Kansas, University of Kentucky, University of Manitoba, University of Michigan, University of Minnesota, University of Notre Dame, University of Oregon, University of Rochester, University of Santa Clara, University of Toronto, Union College, United States Naval Academy, Wake Forest College, Wesleyan University, and Yale University.

The following colleges and universities, in alphabetical order, entered individual contestants only: Albertus Magnus College, Arizona State College (Tempe), Bowling Green State University, Central College, Duke University, Duquesne University, Florida Southern College, Grinnell College, Hope College, Kent State University, Marian College, Park College, Queens College (Flushing, N. Y.), Southeastern State College, The Ohio State University, University of Alaska, University of Ottawa, University of Washington, Ursinus College, Washington University, and West Virginia Wesleyan College.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions, will appear in a later issue of this MONTHLY.

MATHEMATICAL NOTES

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REMARK ON A RECENT NOTE ON LINEAR FORMS

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Suppose that a, d, and s are positive integers and that a and d are coprime. For given integral n consider the solvability in integers $x_0, x_1, \cdots, x_s$ of

$$(1) \qquad n = ax_0 + (a+d)x_1 + \cdots + (a+sd)x_s,\ x_0 \geqq 0, x_1 \geqq 0, \cdots, x_s \geqq 0.$$

Put $N = ga + (d-1)(a-1)$, where g is the smallest integer greater than $(a-2)/s$. Generalizing a classical result [2] for the case $s=1$ and a theorem of Alfred Brauer [1] for the case $d=1$, J. B. Roberts [3] has recently proved:

If $n > N$, then (1) *is solvable. If $n = N-1$, then* (1) *is not solvable.* (In short, $N-1$ is the largest integer for which (1) is not solvable.)

Although Roberts' proof is elementary, it is very involved. Here we present a much simpler proof.

Putting $y_i = \sum_{j=i}^{s} x_j$ for $i = 0, 1, \cdots, s$, we see that the solvability of (1) in integers $x_0, x_1, \cdots, x_s$ is equivalent to the solvability in integers $y_0, y_1, \cdots, y_s$ of

$$(2) \qquad n = ay_0 + d(y_1 + \cdots + y_s),\ y_0 \geqq y_1 \geqq \cdots \geqq y_s \geqq 0.$$

Now for a given y_0 the integers expressible in the form $y_1 + \cdots + y_s$ with $y_0 \geqq y_1 \geqq \cdots \geqq y_s$ are precisely the integers z such that $0 \leqq z \leqq sy_0$. Thus the solvability of (2) in integers $y_0, y_1, \cdots, y_s$ is equivalent to the solvability in integers y, z of

$$(3) \qquad n = ay + dz, \qquad 0 \leqq z \leqq sy.$$

Accordingly it suffices to discuss the solvability of (3).

First suppose $n \geqq N$. Since a and d are coprime there exists an integer z such that $dz \equiv n \pmod{a}$ and $0 \leqq z \leqq a-1$. Hence $n - dz = ay$, where y is an integer. Further

$$ay = n - dz \geqq n - d(a-1) \geqq N - d(a-1) = ga - (a-1) > (g-1)a.$$

Thus $y > g-1$; that is, $y \geqq g$. By assumption $sg > a-2$; that is, $sg \geqq a-1$. Consequently

$$sy \geqq sg \geqq a - 1 \geqq z.$$

Thus (3) is solvable and therefore (1), so that the first assertion of Roberts' theorem is proved.

Now let $n = N-1$. Suppose y and z are integers such that $n = ay + dz$ and

$z \geqq 0$. Since $N-1=a(g-1)+d(a-1)$, we have $z \equiv a-1 \pmod{a}$. Hence $z \geqq a-1$ and $y \leqq g-1$, so that

$$sy \leqq s(g-1) \leqq a-2 < a-1 \leqq z.$$

Thus (3) and (1) are not solvable in this case and Roberts' theorem is completely proved.

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THE MATRIX EQUATION $X^2-I=0$ OVER A FINITE FIELD*

JOHN H. HODGES, University of Buffalo

1. Introduction and notation. Let $GF(q)$ denote the finite field of order q. In this paper we consider the problem of determining the number $N=N\{x^2-1,\ m\}$ of $m \times m$ matrices X over $GF(q)$ such that $X^2-I=0$. The more general problem of determining the number $N\{E(x), m\}$ of $m \times m$ matrices X over $GF(q)$ which satisfy the equation $E(X)=0$, where $E(x)$ is a polynomial with scalar coefficients over $GF(q)$, will be treated in a later paper.

Except as indicated, Roman capitals $A, B, \cdots$ will denote square matrices of order m over $GF(q)$. I_t will denote the identity matrix of order t. We will need the following well-known formula for the number g_t of nonsingular matrices of order t over $GF(q)$:

$$g_t = q^{t^2} \prod_{i=1}^{t} (1 - q^{-i}) = \prod_{i=0}^{t-1} (q^t - q^i). \tag{1.1}$$

We shall consider separately the two cases where q is odd and even in Section 2 and 3 respectively.

2. $X^2-I=0$ for q odd. Let $N=N_0$ for q odd. If X is an $m \times m$ matrix which satisfies the equation $X^2-I=0$, since the minimum polynomial of X must divide $x^2-1=(x-1)(x+1)$, the elementary divisors of $xI-X$ are of the form $x \pm 1$. If $xI-X$ has t elementary divisors of the form $x-1$ and $m-t$ of the form $x+1$, then it follows ([2] p. 241) that X is similar to $J_t=\text{diag}\,(I_t, -I_{m-t})$ with $0 \leqq t \leqq m$. Conversely, it is easily seen that if X is similar to some J_t, then $X^2-I=0$. Thus X is a solution of the equation if and only if it is similar to some J_t, $0 \leqq t \leqq m$.

Let $S_0(m, t)$ be the number of distinct $m \times m$ matrices X which are similar to J_t. Since no matrix X is similar to more than one such J_t it follows that

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$$N_0\{x^2 - 1, m\} = \sum_{t=0}^{m} S_0(m, t). \tag{2.1}$$

Now as P runs through all nonsingular matrices of order m, $P^{-1}J_tP$ runs through all $m \times m$ matrices X which are similar to J_t, each X appearing the same number of times, namely as many times as J_t itself appears. But if $P^{-1}J_tP = J_t$, then P is a nonsingular matrix which commutes with J_t. Therefore the number of times each matrix X which is similar to J_t appears as the value of the form $P^{-1}J_tP$ is the number $C_0(m, t)$ of nonsingular matrices P of order m which commute with J_t. Recalling that g_m is the number of nonsingular matrices of order m and that $S_0(m, t)$ is the number of $m \times m$ matrices X which are similar to J_t, it follows that we have

$$g_m = S_0(m, t)C_0(m, t) \qquad 0 \leq t \leq m. \tag{2.2}$$

It can easily be shown by a direct calculation that

$$C_0(m, t) = g_t g_{m-t}. \tag{2.3}$$

Then by substituting (2.3) into (2.2), solving for $S_0(m, t)$ and substituting into (2.1) we get

THEOREM 1. *The number of $m \times m$ matrices X over $GF(q)$, q odd, satisfying the equation $X^2 - I = 0$ is*

$$N_0\{x^2 - 1, m\} = g_m \sum_{t=0}^{m} \frac{1}{g_t g_{m-t}}, \tag{2.4}$$

where g_t is given by (1.1) *for $0 < t \leqq m$ and $g_0 = 1$.*

Using the q-binomial coefficients defined by

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{(1 - q^m) \cdots (1 - q^{m-r+1})}{(1 - q) \cdots (1 - q^r)} \text{ for } 0 < r \leqq m, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1, \tag{2.5}$$

we are able to give an interesting formula for $N_0\{x^2 - 1, m\}$. Substituting for g_m, g_t, g_{m-t} in (2.4) the explicit values given by (1.1) and simplifying gives

$$N_0\{x^2 - 1, m\} = \sum_{t=0}^{m} q^{2t(m-t)} \prod_{k=1}^{m-t} \frac{(1 - q^{-t-k})}{(1 - q^{-k})}, \tag{2.6}$$

where the value of the inner product is taken as 1 when $t = m$. For fixed t, $0 \leqq t < m$, in view of the definition (2.5),

$$\prod_{k=1}^{m-t} \frac{(1 - q^{-t-k})}{(1 - q^{-k})} = q^{-t(m-t)} \prod_{k=1}^{m-t} \frac{(1 - q^{m+1-k})}{(1 - q^k)} = q^{-t(m-t)} \begin{bmatrix} m \\ m - t \end{bmatrix}. \tag{2.7}$$

Finally, substituting (2.7) into (2.6) we get

$$N_0\{x^2-1, m\} = \sum_{t=0}^{m} q^{t(m-t)} \begin{bmatrix} m \\ m-t \end{bmatrix}. \tag{2.8}$$

3. $X^2-I=0$ for q even. Let $N=N_e$ for q even. In this case, if X is an $m\times m$ matrix which satisfies the equation $X^2-I=0$, since the minimum polynomial of X must divide $x^2-1=(x-1)^2$, the elementary divisors of $xI-X$ are of the form $x-1$ or $(x-1)^2$. If $xI-X$ has t elementary divisors of the second form and $m-2t$ of the first, then X is similar to $H_t=\text{diag}\,(I_{m-2t}, E_1, \cdots, E_t)$ where $E_i=E$ for all i and E is the companion matrix of $(x-1)^2$. Thus we see that X is a solution of the equation if and only if it is similar to some H_t, $0\leqq 2t\leqq m$.

Proceeding as in the first case, let $S_e(m, t)$ be the number of distinct $m\times m$ matrices X which are similar to H_t. Then we have

$$N_e\{x^2-1, m\} = \sum_{0\leq 2t\leq m} S_e(m, t). \tag{3.1}$$

Also as before, if $C_e(m, t)$ is the number of nonsingular matrices of order m which commute with H_t, then

$$g_m = S_e(m, t)C_e(m, t). \tag{3.2}$$

Now, L. E. Dickson ([1] p. 235) has given a somewhat complicated formula for the number of nonsingular matrices of order m commuting with a given matrix A of order m, in terms of the degrees of the prime factors of the minimum polynomial of A and the number and exponents of the various powers of these prime factors which are elementary divisors of A. We will not take the space to state this formula here, but using it we find that the number of nonsingular matrices of order m which commute with H_t, for $0\leqq 2t\leqq m$, is

$$C_e(m, t) = q^{t(2m-3t)} g_t g_{m-2t}, \tag{3.3}$$

where g_t is defined by (1.1) for $0<t\leqq m$ and $g_0=1$. Therefore, using (3.3), (3.2) and (3.1), we get

THEOREM 2. *The number of $m\times m$ matrices X over $GF(q)$, q even, satisfying the equation $X^2-I=0$ is*

$$N_e\{x^2-1, m\} = g_m \sum_{0\leq 2t\leq m} \frac{q^{-t(2m-3t)}}{g_t g_{m-2t}}, \tag{3.4}$$

where g_t is given by (1.1) *for $0<t\leqq m$ and $g_0=1$.*

The author's investigations have not disclosed a representation for N_e corresponding to the representation of N_0 given by (2.8).

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AN ELEMENTARY PROOF OF THE ERDÖS-MORDELL THEOREM

LEON BANKOFF, Los Angeles, California

The Erdös-Mordell Theorem states that *if O is an arbitrary point within a triangle ABC and P, Q, R are the feet of the perpendiculars from O upon sides BC, CA, AB respectively, then* $OA+OB+OC \geqq 2(OP+OQ+OR)$.

A request by Paul Erdös for a proof of this theorem appeared as Problem 3739 of this MONTHLY (June–July 1935, p. 396), and almost two years later (April 1937, pp. 252–254) a brief trigonometric solution by L. J. Mordell was published, along with a considerably longer trigonometric solution by D. F. Barrow, who, in addition, showed that the equality sign holds only when triangle ABC is equilateral and O is the centroid of the triangle. A discussion of the theorem by L. F. Tóth in 1953 ([1] pp. 12–14, 28) induced the search for a simple and elementary proof of the theorem by the methods of synthetic geometry. Such a proof has recently been supplied by D. K. Kazarinoff [2], and this proof has been extended by N. D. Kazarinoff [3] to establish an analogous theorem about the tetrahedron. An appropriate analog for the theorem in Euclidean n-space, $n>3$, has not yet been formulated.*

The following synthetic proof of the Erdös-Mordell Theorem seems simpler and more elementary than that of D. K. Kazarinoff.

Let P_1P_2, Q_1Q_2, R_1R_2 denote the respective orthogonal projections of RQ, PR, QP on BC, CA, AB. Then

$$(1)\qquad OA + OB + OC \geqq OA(P_1P + PP_2)/RQ + OB(Q_1Q + QQ_2)/RP + OC(R_1R + RR_2)/PQ.$$

Because of the equality of angles ROB and RPP_1 in the cyclic quadrilateral $BPOR$, the right triangles PRP_1 and OBR are similar and $P_1P = RP \cdot OR/OB$. In a like manner we find $PP_2 = PQ \cdot OQ/OC$; $Q_1Q = PQ \cdot OP/OC$; $QQ_2 = RQ \cdot OR/OA$; $R_1R = RQ \cdot OQ/OA$ and $RR_2 = RP \cdot OP/OB$.

Substituting in (1) and grouping terms with respect to the factors OP, OQ, OR, we obtain

$$OA + OB + OC \geqq OP\left(\frac{RP \cdot OC}{PQ \cdot OB} + \frac{PQ \cdot OB}{RP \cdot OC}\right) + OQ\left(\frac{PQ \cdot OA}{RQ \cdot OC} + \frac{RQ \cdot OC}{PQ \cdot OA}\right) + OR\left(\frac{RP \cdot OA}{RQ \cdot OB} + \frac{RQ \cdot OB}{RP \cdot OA}\right) \geqq 2(OP + OQ + OR).$$

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* The references given in the second paragraph were furnished by the referee.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

TERSE TRIGONOMETRY

C. M. FULTON, University of California, Davis

This is an attempt at formulating a new approach to trigonometry. Its conciseness and rigor should make it suitable for good students. The basic idea is a truly general definition of the trigonometric functions.

Let $P(x, y)$, $x^2+y^2=1$, be on the terminal side of an angle θ and $P'(x', y')$, $x'^2+y'^2=1$, on its initial side. The angle with its vertex at the origin O is considered positive or negative according as the rotation involved is counterclockwise or clockwise. We define

$$\cos\theta = xx' + yy', \qquad \sin\theta = yx' - xy'. \tag{1}$$

Interchanging the sides of the angle it is seen that

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta. \tag{2}$$

The following identities are easily established:

$$(xx' + yy')^2 + (yx' - xy')^2 = 1, \tag{3}$$

$$(x - x')^2 + (y - y')^2 = 2 - 2(xx' + yy'). \tag{4}$$

The meaning of (3) is obvious. Identity (4), on the other hand, is crucial in that it shows the invariance of the definition of cosine: if the angle is rotated, the distance PP' remains unchanged and so does the cosine. Hence, because of (3) the same is true for the square of the sine. Moreover, a continuity argument will take care of the sine itself. It should be apparent to the reader that, in order to save space, we are using technical rather than pedagogical language in this discussion.

We now use the invariance demonstrated above and place the angle in standard position letting $x'=1$, $y'=0$. The definitions (1) reduce to

$$\cos\theta = x, \qquad \sin\theta = y. \tag{5}$$

Let us remind ourselves at this stage that the much-stressed trigonometric identities involving one angle only could all be written as algebraic identities by using (5). Returning to definitions (1) we denote by α and α' the angles in standard position whose terminal sides are OP and OP', respectively. Then θ and $\alpha-\alpha'$ differ by some integral multiple of 360° and with the aid of (5) we obtain the formulas for $\cos(\alpha-\alpha')$ and $\sin(\alpha-\alpha')$. On account of (2) the corresponding formulas for $\alpha+\alpha'$ are easily derived. Eventually, the so-called

reduction formulas follow as special cases. All these derivations are, of course, completely general.

Clearly, the definitions (5) can be changed to the usual form $\cos\theta = x/r$, $\sin\theta = y/r$ by taking a point P at a distance r from the origin. This leads directly to the solution of right triangles. Furthermore, identity (4) is generalized to

$$(x - x')^2 + (y - y')^2 = r^2 + r'^2 - 2rr' \cos\theta,$$

which is precisely the law of cosines. May we be allowed to suggest at this point that for practical purposes the latter be rewritten as a haversine formula, namely

$$c^2 = (a - b)^2 + 4ab \operatorname{hav} \gamma.$$

Finally, to find a trigonometric formula for the area of triangle $OP'P$, let $x' = r'$, $y' = 0$, $y > 0$. Then the area $K = \frac{1}{2}r'y = \frac{1}{2}r'r \sin\theta$. We can now prove the law of sines without difficulty, using the formula for K.

GENERAL SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

R. D. Larsson, Clarkson College of Technology

Texts on ordinary differential equations define linearity and develop the properties of the linear differential equations, but then fail to use these properties to the maximum extent possible.

Consider the homogeneous differential equation with constant coefficients

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0.$$

Assuming a solution of the form $y = e^{mx}$ we obtain roots of the auxiliary equation $m = m_1, m_2$ and the general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ if $m_1 \neq m_2$.

Using the property that linear combinations of solutions are solutions,

$$y = \frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$$

is a solution.

$$y = \lim_{m_1 \to m_2} \frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$$

is a solution for $m_1 = m_2$.

Using L'Hôpital's Rule

$$y = \lim_{m_1 \to m_2} \frac{xe^{m_1 x}}{1} = xe^{m_2 x}$$

is a solution, and $y = c_1 x e^{m_2 x} + c_2 e^{m_2 x}$ is the general solution.

Consider Euler's differential equation $a_2 x^2 y'' + a_1 x y' + a_0 y = 0$. Assuming a solution of the form $y = x^m$ we obtain roots of the auxiliary equation as $m = m_1, m_2$ and the general solution is $y = c_1 x^{m_1} + c_2 x^{m_2}$ if $m_1 \neq m_2$.

Again using the property that linear combinations of solutions are solutions,

$$y = \frac{x^{m_1} - x^{m_2}}{m_1 - m_2}$$

is a solution.

$$y = \lim_{m_1 \to m_2} \frac{x^{m_1} - x^{m_2}}{m_1 - m_2}$$

is a solution for $m_1 = m_2$.

Using L'Hôpital's Rule

$$y = \lim_{m_1 \to m_2} \frac{x^{m_1} \ln x}{1} = x^{m_2} \ln x$$

is a solution. $y = c_1 x^{m_2} \ln x + c_2 x^{m_2}$ is the general solution.

Why not use these basic principles to obtain the results above instead of a different device for each type of equation?

But to extend this principle even further, consider the second order differential equation with a regular singular point at $x = 0$.

$$a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = 0.$$

Using the method of Frobenius we assume a solution of the form $y = \sum_{n=0}^{\infty} b_n x^{n+k} = F(x, k)$.

We get roots of the indicial equation $k = k_1, k_2$. Assume $k_2 \geqq k_1$.

Case 1. $k_2 = k_1$. If this does not lead to the general solution, suppose $\lim_{k \to k_2} F(x, k)$ is a solution.

By the property of linearity

$$y = \lim_{k \to k_2} \frac{F(x, k) - F(x, k_2)}{k - k_2}$$

is a solution. Therefore

$$y = \lim_{k \to k_2} \frac{\dfrac{\partial F(x, k)}{\partial k}}{1}$$

is a solution, and

$$y = c_1 \lim_{k \to k_2} \frac{\partial F(x, k)}{\partial k} + c_2 F(x, k_2)$$

is the general solution.

Case II. $k_2 > k_1$. k_2 will give a particular solution. If k_2 differs from k_1 by an integer, k_1 may then give another solution, or $\lim_{k \to k_1} (k - k_1)F(x, k) = F(x, k_2)$.

$$y = \lim_{k \to k_1} \frac{(k - k_1)F(x, k) - F(x, k_2)}{k - k_1}$$

is a solution, or

$$y = \lim_{k \to k_1} \frac{\dfrac{\partial}{\partial k}\left[(k - k_1)F(x, k)\right]}{1},$$

and

$$y = c_1 \lim_{k \to k_1} \frac{\partial}{\partial k}\left[(k - k_1)F(x, k)\right] + c_2 F(x, k_2).$$

Under Case II, for the solution of Bessel's differential equation, where $y = J_n(x)$ with n an integer and $J_n(x) = (-1)^n J_{-n}(x)$, we write

$$y_n(x) = \lim_{n \to \text{integer}} \frac{(\cos n\pi)J_n(x) - J_{-n}(x)}{\sin n\pi}$$

to obtain the solution of the second kind. This again can be evaluated by L'Hôpital's Rule.

My thesis is that if one can use the same property throughout the work it is easier for the student to follow, and to remember, the basic principles. Trick devices are interesting but do not lend themselves to an extension of the theory. The above method may be used when two particular solutions become identical in the limit and when the equation is homogeneous and linear.

ON THE SLOPES OF PERPENDICULAR LINES

S. LEADER, Rutgers University

The usual proof that the slopes of lines which meet at right angles are negative reciprocals is based upon a trigonometric identity. A geometric proof can be given using the theorem that the interior altitude of a right triangle is the geometric mean of the segments into which it divides the hypotenuse.

Let O be the intersection of the lines l_1 and l_2 which meet at right angles. We exclude the case in which either line is vertical by assuming both lines have slopes. Draw a horizontal segment OA extending one unit to the right of O. Draw a vertical line through A meeting l_1 at B_1 and l_2 at B_2. Then the slopes m_1 and m_2 are given respectively by the directed vertical segments AB_1 and AB_2. Since these segments have opposite directions, the product m_1m_2 is negative. Now OA is the interior altitude of the right triangle B_1OB_2. Therefore, $|OA|^2 = |AB_1|\,|AB_2|$. That is, $1 = |m_1m_2|$. Finally, since m_1m_2 is negative, $m_1m_2 = -1$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1326. *Proposed by P. L. Chessin, University of Maryland*

With n straight line segments of lengths 1, 2, 3, $\cdots$, n, how many non-degenerate triangles can be constructed?

E 1327. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

A confirmed angle-watcher marked the angle α formed by the hands of a clock. Some time later he noticed that the hands trisected α. In how short a time could this have happened? How soon after 3:00 could one start such an experiment?

E 1328. *Proposed by Winton Laubach, Colorado School of Mines*

A ray from the origin intersects the circle $\rho=1$ at C and the spiral $\rho=e^{\theta}$, $\theta>0$, at S. Tangents to the circle at C and to the spiral at S intersect at P. Identify the locus of P.

E 1329. *Proposed by Jose Gallego-Diaz, Vanderbilt University*

A spherical square is a spherical quadrilateral whose four sides are equal and whose four angles are equal. If we let a, b, c denote the areas of the spherical squares constructed on the legs and hypotenuse of a right spherical triangle, show that

$$\tanh^{-1}(\sin c/4) = \tanh^{-1}(\sin b/4) + \tanh^{-1}(\sin a/4).$$

E 1330. *Proposed by J. L. Pietenpol, Columbia University*

Find the limit of the infinite product $\prod_{n=1}^{\infty}(1+1/a_n)$, where $a_1=1$, $a_n=n(a_{n-1}+1)$.

SOLUTIONS

A Generalization of E 1242

E 1296 [1958, 42]. *Proposed by Beckham Martin, Owens-Illinois Glass Co., Toledo, Ohio*

Problem E 1242 [1957, 433] says that the circle orthogonal to the circles (A'), (B'), (C') inscribed in the squares constructed exteriorly (or interiorly) on the sides of a triangle ABC is concentric with the nine point circle of triangle $A'B'C'$. Show, more generally, that all circles cutting the circles (A'), (B'), (C') under angles α, β, γ, respectively, where

$$\cos\alpha \sin A = \cos\beta \sin B = \cos\gamma \sin C,$$

are concentric with the nine point circle of triangle $A'B'C'$.

Solution by Victor Thébault, Tennie, Sarthe, France. Let a, b, c denote the sides BC, CA, AB of triangle ABC, R the circumradius of triangle ABC, N' the nine point center of triangle $A'B'C'$, (ω) the circle with center ω and radius r which cuts (A'), (B'), (C') under the angles α, β, γ such that

$$\cos\alpha \sin A = \cos\beta \sin B = \cos\gamma \sin C = k,$$

and M one of the points of intersection of (A') and (ω). By the law of cosines, applied to triangle $A'\omega M$, we have

$$(A'\omega)^2 = (\omega M)^2 + (A'M)^2 + 2(A'M)(\omega M)\cos\alpha,$$

which gives

$$(A'\omega)^2 - a^2/4 = r^2 + ar\cos\alpha = r^2 + 2Rr\cos\alpha\sin A = r^2 + 2kRr.$$

In this way we find

$$(1) \qquad (A'\omega)^2 - a^2/4 = (B'\omega)^2 - b^2/4 = (C'\omega)^2 - c^2/4 = r^2 + 2kRr.$$

Now, if r' denotes the radius of the circle having center N' and orthogonal to (A'), (B'), (C'), we have

$$(2) \qquad (A'N')^2 - a^2/4 = (B'N')^2 - b^2/4 = (C'N')^2 - c^2/4 = r'^2.$$

Subtracting (2) from (1) we find

$$(A'\omega)^2 - (A'N')^2 = (B'\omega)^2 - (B'N')^2 = (C'\omega)^2 - (C'N')^2,$$

from which it follows that ω coincides with N'.

Also solved by D. R. Brillinger and D. C. B. Marsh.

Concerning the Product of a Number by Its Reversal

E 1297 [1958, 42]. *Proposed by D. C. B. Marsh, Colorado School of Mines*

In Solution II of Problem E 1243 [1957, 434] occurs the following conjecture for proof or disproof: When an integer and its reversal are unequal, their product is never a square except when both are squares.

Show that for any $n>2$ there is a nonsymmetric, nonsquare, n digit integer whose product with its reversal is a square.

Solution by Joe Lipman, University of Toronto. If n is odd we may take the number as $2(10^{k-1}+2)^2$, where $k=(n+1)/2$; if n is even we may take the number as $11(10^{k-1}+2)^2$, where $k=n/2$.

Also solved by Merrill Barnebey, D. M. Brown, Michael Goldberg, Naoki Kimura, M. S. Klamkin, W. M. McKeeman, Otto Mond, C. S. Ogilvy, Benjamin Sapolsky, W. B. Stovall, Jr., and the proposer.

Editorial Note. Many solvers gave numbers with unorthodox reversals, such as $2(10^{2k})$ and $11(10^{2k})$.

An Infinite Nesting of Parentheses

E 1299 [1958, 43]. *Proposed by P. L. Chessin, Westinghouse Electric Corporation*

Find the limit of $1+x(1-x[1+x(1-x[1+\cdots])])$, for $|x|<1$.

I. *Solution by T. H. Slook, Temple University.* Removal of parentheses gives

$$1 + x - x^2 - x^3 + x^4 + x^5 - \cdots,$$

which converges absolutely for $|x|<1$. We may then regroup terms to obtain

$$(1 + x)(1 - x^2 + x^4 - x^6 + \cdots),$$

which, for $|x|<1$, represents $(1+x)/(1+x^2)$.

II. *Solution by Joe Lipman, University of Toronto.* The given series is absolutely convergent for $|x|<1$, therefore it has a limit L, which then must satisfy $L=1+x(1-xL)$. Hence $L=(1+x)/(1+x^2)$.

Also solved by J. L. Alperin, A. G. Anderson, Philip Bacon, Edward Barbeau, Merrill Barnebey, A. P. Boblétt, Julian Braun, D. A. Breault, D. R. Brillinger, E. W. Brown, C. N. Campopiano, A. E. Danese, T. W. Daniel, J. E. Darraugh, R. S. Dinsmore, E. S. Eby, S. J. Einhorn, E. L. Ellis and D. L. Muench (jointly), G. V. Emerson, G. W. Erwin, W. V. Gamzon, H. M. Gehman, Michael Goldberg, Ronald Gordon, R. E. Graf, Donald Greenstein, Emil Grosswald, J. W. Haake, R. H. Hou, A. R. Hyde, N. S. Kandalgaonkar, Seymour Kass, M. A. Kirchberg, M. S. Klamkin, J. D. E. Konhauser, Morton Kupperman, P. S. Landweber, R. L. London, W. M. McKeeman, D. C. B. Marsh, C. T. Molloy, Jr., Otto Mond, Morris Morduchow, J. B. Muskat, D. E. Myers, C. S. Ogilvy, F. D. Parker, D. J. Peterson, C. F. Pinzka, L. A. Ringenberg, Jeff Ritterman, D. A. Robinson, H. D. Ruderman, D. L. Schell, Francis Sevier, Paul Shaefer, Arnold Singer, D. L. Smith, O. E. Stanaitis, W. B. Stovall, Jr., Donato Teodoro, C. W. Trigg, Chih-yi Wang, Walter Wiebenson, B. G. Willis, Clement Winston, David Zeitlin, and the proposer.

A Class of Determinants

E 1300 [1958, 43]. *Proposed by D. A. Robinson, University of Wisconsin*

Let $D_1, D_2, D_3 \cdots$ be the determinants

$$|1|, \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{vmatrix}, \cdots,$$

respectively. Find the value of D_n for any positive integer n.

Solution by A. R. Erskine, Pennsylvania State University. D_n may be described as follows: The first row consists of $n+1$ ones followed by $n-2$ zeros. The next $n-2$ rows are the cyclic permutations of this row which introduce successively 1, 2, $\cdots$, $n-2$ zeros on the left. The nth row consists of the integers from 1 to n followed by $n-1$ zeros. The remaining $n-1$ rows are the corresponding cyclic permutations of the nth row.

To evaluate D_n, from row $k+n-1$ subtract the sum of the rows indexed from k to $k+n-2$ inclusive, giving k successively the values 1, 2, $\cdots$, n. This will produce a triangular matrix. The first n elements on the main diagonal will be 1; the last $n-1$ elements will be $n+1$. Hence D_n has the value $(n+1)^{n-1}$.

Also solved by A. G. Anderson, Edward Barbeau, Merrill Barnebey, W. J. Blundon, D. R. Brillinger, E. W. Brown, A. E. Danese, T. W. Daniel, J. W. Gammill, W. V. Gamzon, Michael Goldberg, Emil Grosswald, J. W. Haake, B. A. Hausmann, S.J., M. S. Klamkin, J. D. E. Konhauser, Morton Kupperman, Joe Lipman, D. C. B. Marsh, Otto Mond, C. S. Ogilvy, Jeff Ritterman, Benjamin Sapolsky, Nathan Schwid, O. E. Stanaitis, C. W. Trigg, L. H. Tulloch, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4798. *Proposed by John McCarthy, Dartmouth College*

Let $\alpha_1, \cdots, \alpha_n$ be algebraic numbers linearly independent over the rationals.

Show that there is a positive constant C and an integer N such that if $m_1, \cdots, m_n$ are rational integers not all zero then

$$|m_1\alpha_1 + \cdots + m_n\alpha_n| \geqq \frac{C}{\{|m_1| + \cdots + |m_n|\}^N}.$$

This generalizes Liouville's theorem on the approximation of algebraic numbers by rationals.

4799. *Proposed by K. L. Chung, Syracuse University*

Find a function continuous in $(0, 1]$ without any interval of constancy which takes on rational values almost everywhere.

4800. *Proposed by N. S. Mendelsohn, The University of Manitoba*

Let $\{u_n\}$ be a sequence with a recurrence formula of the form

$$u_{n+1} = a_0u_n + a_1u_{n-1} + \cdots + a_ku_{n-k},$$

where $a_0, \cdots, a_k$ are real constants. Show that the sequence obtained by taking every rth term of this sequence satisfies a recurrence of the same length; *i.e.*, if $v_n = u_{rn}$ for $n = 1, 2, \cdots$, there exist constants $A_0, \cdots, A_k$ such that for $n = k+1, k+2, \cdots$,

$$v_{n+1} = A_0v_n + A_1v_{n-1} + \cdots + A_kv_{n-k}.$$

4801. *Proposed by S. W. Golomb, Pasadena, California*

Let $R(n)$ denote the minimum number of terms required in representing n as a sum of squares. If n is restricted to perfect numbers, show that

$$R(n) = \begin{cases} 2 & \text{for odd } n, \text{ if any;} \\ 3 & \text{for } n = 6; \\ 4 & \text{for even } n \neq 6. \end{cases}$$

4802. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let $A = (a_{ij})$ be a real symmetric matrix of order $n(\geqq 3)$ and of rank $\leqq 3$. If A is positive semi-definite and if $a_{ii} = 1$ $(1 \leqq i \leqq n)$, then

$$\operatorname*{Max}_{i \neq j} a_{ij} \geqq \tfrac{1}{2} \operatorname{cosec}^2 \omega_n - 1,$$

where $\omega_n = n\pi/6(n-2)$.

4803. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Let $\sum a_nx^n$ be analytic at 0 and suppose $a_n > 0$, $a_{n+1}a_{n-1} > a_n^2$. Prove that the expansion $1/\sum a_nx^n$ has all negative coefficients (except for the constant term).

SOLUTIONS

Feuerbach Hyperbola Tangent to Circumcircle

4751 [1957, 509]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the Feuerbach hyperbola of a triangle ABC, with orthocenter H and incenter I, is tangent to the circle ABC, then it is also tangent to one of the circles BCH, CAH, ABH, and its focal axis is the perpendicular bisector of IH. (The Feuerbach hyperbola is the equilateral hyperbola determined by the four points, A, B, C, and I.)

Solution by the proposer. Since the Feuerbach hyperbola $\mathfrak{H}$ passes through A, B, C, it must be tangent to circle ABC at one of these points, say A. Now, under the isogonal transformation set up by triangle ABC, an equilateral hyperbola corresponds to some line through the circumcenter O of ABC. It follows that $\mathfrak{H}$, being an equilateral hyperbola passing through the fixed point I, must correspond to the line OI. Since the isogonal conjugate of the ray through A tangent to circle ABC at A is parallel to BC, it follows that OI is parallel to BC.

Under the isogonal transformation set up by triangle HBC, hyperbola $\mathfrak{H}$, since it also passes through H, corresponds to a diametral line of circle HBC. This line must have the same orientation as OI, and hence is parallel to BC. It now follows that $\mathfrak{H}$ is tangent to circle HBC at point H.

By elementary geometry it is easy to show that the tangent to circle HBC at H is parallel to the tangent to circle ABC at A. Hence A and H are diametrically opposite points on $\mathfrak{H}$, and the center of $\mathfrak{H}$ is the midpoint ω of AH. Since the center of $\mathfrak{H}$ is the internal Feuerbach point of triangle ABC, we see that $\omega I = r$, the radius of the inscribed circle of triangle ABC. Also, if M is the midpoint of BC, $AH = 2OM = 2r$. It follows that $\omega I = \omega H$, and the focal axis of $\mathfrak{H}$ is thus the perpendicular bisector of IH.

Also solved by J. W. Clawson, who used complex coordinates.

Editorial Note. The figure is rich. One may show, among other things, that AIH is a right triangle, $\mathfrak{H}$ is tangent to OI at I, $OI = (b-c)/2$, $r = R\cos A$ (where R is the circumradius of triangle ABC), $8Rr = 4R^2 - (b-c)^2$, $\cos B + \cos C = 1$. For an analytical study of the isogonal transformation, see, *e.g.*, D. M. Y. Sommerville, *Analytical Conics*, London, 1924.

Quadrics with Common Points

4754 [1957, 596]. *Proposed by T. G. Room, University of Sydney, Australia*

Given the four quadrics

$$L \equiv x^2 + dyz - cyt + bzt = 0, \qquad M \equiv y^2 + dzx + cxt - azt = 0,$$

$$N \equiv z^2 + dxy - bxt + ayt = 0, \qquad K \equiv t^2 - ayz - bzx - cxy = 0.$$

Show that if $1 + a^3 + b^3 + c^3 + d^3 + 3abcd = 0$ then they have six points in common, and otherwise, none.

Solution by the proposer. If the four quadrics have six common points, then, included in the triply infinite (∞^3) system

$$\phi \equiv \alpha L + \beta M + \gamma N + \delta K = 0,$$

there is a net (linear ∞^2 system) of which the base is the unique twisted cubic which passes through the six points. Conversely, if there exists such a net, since ϕ is completely determined by this net and one other quadric, and this quadric meets the twisted cubic in six points, it follows that all the quadrics pass through these six points.

Write $\Delta = 1 + a^3 + b^3 + c^3 + d^3 + 3abcd$,

$$A = a^2 + bcd, \qquad B = b^2 + cad, \qquad C = c^2 + abd, \qquad D = d^2 + abc.$$

It is to be proved that the linear ∞^2 system ψ determined in ϕ by the parameters which satisfy

$$A\alpha + B\beta + C\gamma + D\delta = 0$$

consists, if $\Delta = 0$, of quadrics with a common twisted cubic, and the main theorem is then proved.

All quadrics of the pencil (linear ∞^1 system)

$$\alpha L + \beta M + \gamma N = 0, \qquad A\alpha + B\beta + C\gamma = 0$$

contain the line $x/a = y/b = z/c$, and meet therefore residually in a twisted cubic. This line meets a general quadric of ψ in the two points $(a, b, c, \pm d')$, where

$$d'^2 = d^2 + 4abc.$$

We find similarly for ψ altogether eight base points, namely

$$(a, b, c, \pm d'), \quad (\pm a', b, -c, d), \quad (-a, \pm b', c, d), \quad (a, -b, \pm c', d).$$

It is easily proved that in general no four of these points are coplanar, so that the base of the system ψ is either this set of 8 (associated) points, or a twisted cubic. None of these 8 points lies on any other quadric of ϕ, so that unless ψ has a base twisted cubic, the four original quadrics have no common points.

The system ψ will have a twisted cubic for base only if there is a cone of the system with its vertex at each base point of ψ, and it will be sufficient if we can find cones for two such points.

The point (a, b, c, d') is the vertex of a cone in the system ψ if

$$\begin{aligned}
\alpha 2a + \beta c(d + d') + \gamma b(d - d') &= 0 \\
\alpha c(d - d') + \beta 2b + \gamma a(d + d') &= 0 \\
\alpha b(d + d') + \beta a(d - d') + \gamma 2c &= 0 \\
\delta 2d' &= 0 \\
\alpha A + \beta B + \gamma C + \delta D &= 0.
\end{aligned}$$

The last equation is linearly dependent on the others (with $\delta = 0$ and multipliers $a, b, c, 0, -2$); the first four can be solved for $\alpha, \beta, \gamma, \delta$, if

$$\begin{vmatrix} 2a & c(d+d') & b(d-d') \\ c(d-d') & 2b & a(d+d') \\ b(d+d') & a(d-d') & 2c \end{vmatrix} = 0,$$

i.e., if $abc\Delta = 0$.

In the same way we find that each of the 8 points is the vertex of a cone in the system ψ if $\Delta = 0$. Two of these cones meet, apart from the line joining their vertices, in a twisted cubic which passes through all 8 points, and which therefore lies on all quadrics of ψ.

A Permutation Problem

4755 [1957, 596]. *Proposed by Chandler Davis, Institute for Advanced Study*

In how many ways can the first n positive integers be arranged in alternately increasing and decreasing order? That is, how many permutations $\pi: \pi(1), \cdots, \pi(n)$ are there such that the quantities $(-1)^k\{\pi(k+1)-\pi(k)\}$, for $k=1, \cdots, n-1$ have all the same sign?

Solution by W. J. Blundon, Memorial University of Newfoundland. Let P_n be the required number of arrangements of the first n positive integers, under the restriction that the common sign of the stated quantities is negative. The integer n, being the largest in the set, is necessarily of the form $\pi(2i)$, $i=1, 2, \cdots, [n/2]$. The integers to the left of n can be chosen in $\binom{n-1}{2i-1}$ ways, and each such selection can be arranged in P_{2i-1} ways. The integers to the right of n can be arranged in P_{n-2i} ways. Hence

$$P_n = \sum_{i=1}^{[n/2]} \binom{n-1}{2i-1} P_{2i-1}P_{n-2i}, \qquad n = 1, 2, \cdots,$$

where, for convenience, we define $P_0 = 1$. Putting $P_n = n!Q_n$, we have

$$Q_0 = 1; \qquad nQ_n = \sum_{i=1}^{[n/2]} Q_{2i-1}Q_{n-2i}, \qquad n = 1, 2, \cdots.$$

Define $f(x) = \sum_{n=0}^{\infty} Q_n x^n$. Then it is easily verified that

$$\{f(x)\}^2 = -1 + 2f'(x).$$

The solution of this differential equation gives

$$\sec x + \tan x = f(x) = \sum_{n=0}^{\infty} P_n x^n/n!.$$

From the well-known expansions of sec x and tan x, we have

$$P_n = \begin{cases} B_n & n \text{ even}, \\ \dfrac{2^{n+1}(2^{n+1}-1)}{n+1} B_n & n \text{ odd}, \end{cases}$$

where the B's are alternately Bernoulli numbers and Euler numbers $(1/6, 1, 1/30, 5, \cdots,)$.

We now remove the restriction of the first sentence of this solution. Then, by symmetry, the required number of arrangements is $2P_n$, (except when $n=1$, when the restriction has no meaning). The number of arrangements for small n is given by

n	1	2	3	4	5	6	7	8	9	10
$2P_n$	1	2	4	10	32	122	544	2770	15872	101042

Also solved by W. H. Furry, E. C. Milner, L. E. Clarke, A. van Heemert, and the proposer.

Editorial Note. Using known expansions for the tangent and secant, Furry gets the result in the form

$$P_n = 2(2/\pi)^{n+1}n!\{1 + (-3)^{-n-1} + 5^{-n-1} + (-7)^{-n-1} + \cdots\}.$$

For large n the convergence is rapid, and the first term provides a good asymptotic expression for P_n.

The proposer notes an obvious connection with his problem no. 4714 [1957, 679–680]. He notes also that, because of the uniqueness of the Taylor expansion, P_n is the nth derivative of $\sec x + \tan x$, evaluated at $x=0$.

Real Functions

4756 [1956, 596]. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $p(x_1, \cdots, x_n)$, $q(x_1, \cdots, x_n)$ be two real functions of n real variables x_i, defined and continuous in a parallelotope R: $0 \leqq x_i \leqq a_i < \infty$. Assume that $p(x_1, \cdots) = q(x_1, \cdots) = 0$ whenever $x_1x_2 \cdots x_n = 0$, and that $p(x_1, \cdots) > 0$, $q(x_1, \cdots) \geqq 0$, when $x_1x_2 \cdots x_n \neq 0$. Prove that there exists a real function $h(u)$ of a real variable u, defined, continuous and strictly increasing for $u \geqq 0$, $h(0) = 0$, such that throughout R

$$h\{q(x_1, \cdots)\} < p(x_1, \cdots).$$

Solution by Neill McShane, Yale University. Let $\mathbf{x} = (x_1, \cdots, x_n)$. Since R is a compact domain, the continuous function $r(\mathbf{x})$ defined implicitly by $q(\mathbf{x}) = r(\mathbf{x})p(\mathbf{x})$ is bounded above uniformly by some number r_0. Assume $q(\mathbf{x})$ not identically 0, since otherwise the problem is trivial. Then $r_0 > 0$. $h(u) = u/2r_0$ satisfies the demands of the problem, for

$$h(q(\mathbf{x})) = q(\mathbf{x})/2r_0 = r(\mathbf{x})p(\mathbf{x})/2r_0 < p(\mathbf{x}).$$

Also solved by J. Horváth and by the proposer.

Identity Related to the Beta- Function

4757 [1957, 596]. *Proposed by O. P. Aggarwal, University of Washington*

Prove for every integer $n \geqq 0$, and for any positive c,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(c+k)^2} = \frac{\Gamma(c)\Gamma(n+1)}{\Gamma(c+n+1)} \sum_{k=0}^{n} \frac{1}{c+k}.$$

Solution by Tien Chi Chen, IBM Research Center, Ossining, N. Y. Both sides of the equation are equal to $-d\beta(c, n+1)/dc$. This is because

$$\beta(c, n+1) = \int_0^1 t^{c-1}(1-t)^n dt = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{c+k},$$

by expanding $(1-t)^n$ and integrating. Differentiating with respect to c and changing sign, we have the left-hand side of the equation. For the right-hand side we obtain

$$\beta(c, n+1) = \frac{n!}{c(c+1)\cdots(c+n)},$$

$$-\frac{d}{dc}\beta(c, n+1) = \frac{n!}{c(c+1)\cdots(c+n)} \sum_{k=0}^{n} \frac{1}{c+k}.$$

Also solved by R. G. Buschman, L. Carlitz, J. E. Darraugh, Emil Grosswald, Irwin Guttman, Peter Henrici, R. D. James, M. S. Klamkin, Margaret M. LaSalle, D. C. B. Marsh, F. D. Parker, A. K. Rajagopal, B. E. Rhoades, D. A. Robinson, S. C. Saunders, E. M. Scheuer, Blagovest Sendov, R. E. Simmons, Chih-yi Wang, Morgan Ward, David Zeitlin, and the proposer.

Editorial Note. As noted by James, a sequence of similar formulas may be derived by comparing higher derivatives of $\beta(c, n+1)$ or by noting, for example, that $\beta(c, n+1)-\beta(c+1, n+1) = \beta(c, n+2)$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

SELECTED MATHEMATICS BOOKS FOR HIGH SCHOOL LIBRARIES

The National High School and Junior College Mathematics Club, Mu Alpha Theta, which is sponsored by the Mathematical Association of America, has prepared a list, containing five sections of about twenty-five dollars valuation each, of selected mathematics books for high school libraries. This list will provide a well-balanced mathematics library on the high school and junior college level. The list is selective rather than extensive. Copies may be secured by sending a self-addressed, stamped envelope to Richard V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, requesting the high school book list.

Reason and Chance in Scientific Discovery. By R. Taton, Translated by A. J. Pomerans. Philosophical Library, New York, 1957. xii+171 pp. $10.00.

Korzybski in his general semantics defines plants as energy binders, animals as space binders, and man as a time binder. One is tempted to extend this epigrammatic definition to that rare subspecies of Homo sapiens, the geniuses, as idea binders. Taton's book is a survey of the subtlest form of idea-binding: invention and discovery.

What leads to an invention? Can we discover the laws of discovery? These are much-mooted questions, and the answers vary widely. Some writers maintain that "the act of discovery is purely an accident and owes nothing to reason or logic" (Nicolle). This view is rejected by Taton. Especially with regard to mathematical discovery he follows Poincaré and Hadamard in assuming the existence of two more-or-less distinct types of mind: the logical and the intuitional, or the analyst and the geometrician. The former proceeds by sustained and methodical effort, while the latter, as described vividly in Poincaré's self-analysis, may arrive at his results in three stages: first, conscious but often fruitless *concentration* ("stirring of ideas"), followed by a period of rest and *incubation* (subconscious processing of ideas), which stage may end happily by sudden *illumination* and insight, requiring merely verification for its completion.

In the observational and experimental sciences, however, the role of chance in discovery cannot be denied, but even here Taton holds with Pasteur that chance helps those whose minds are well prepared for it. Another point that he develops and illustrates convincingly by historical examples is the fact that a discovery must be attributed not only to its author, but also to its epoch. Scientific discoveries usually appear at the right time and proper background. Premature discoveries are not appreciated by their contemporaries, and even their authors are frequently unable to perceive their full significance. Still another fact: error has occasionally played a paradoxically fruitful role in discovery.

Taton concludes his compact and erudite survey with a word of warning and apprehension regarding the increasingly collective character of modern research. While the advantages of this development are unquestionable, the dangers of putting the accent on the immediately profitable and of reducing individual initiative are undeniable.

The book is handsomely illustrated and the translation reads smoothly, disregarding an occasional solecism like "this phenomena."

PINCUS SCHUB
University of Pennsylvania

An Analytical Calculus. Vol. IV. By E. A. Maxwell. Cambridge University Press, New York, 1957. ix+288 pp. $4.00.

This is the final volume in a series of classroom textbooks by the author covering the calculus and early stages of analysis. The volume under review is divided into three sections: 1) Ordinary differential equations, 2) Functions defined by infinite series and integrals, and 3) La Place and related equations.

The theme of the book is set forth in the first section and it is directed towards enabling the student to develop skill and confidence in solving differential equations. Among the assets of the book are 1) its spirit of liveliness and intellectual honesty, 2) selection of topics, 3) careful statements of the conditions under which the techniques used are correct, 4) numerous examples worked out in detail to illustrate when the techniques succeed or fail, and 5) an abundance of exercises for the student. His proof of theorems, when given, are well within the understanding of the calculus student. The emphasis on the first section is on the nth order differential equation with constant coefficients; in the second section, which is the largest, on a) handling of series including termwise integration and differentiation, b) improper integrals and the differentiation and integration of functions defined by integrals, and c) an introduction to Fourier series; the third section contains an excellent introductory treatment of the La Place, heat, and wave equations, Jacobians, and spherical harmonies. Special functions such as the Bessel, Beta, Gamma, and Legendre functions are discussed. The book is distinctive and well written.

PASQUALE PORCELLI
Illinois Institute of Technology

The Fascination of Numbers. By W. J. Reichmann. Essential Books, Fair Lawn, New Jersey, 1957. 176 pp. $4.00.

This work is intended for the general reader, and purports to show in a simple way the behavior and structure of numbers and their relation to each other.

Many of the recreational aspects of numbers are discussed, magic squares, puzzles, Russian multiplication and so on. Many of the famous problems accessible to the lay reader are presented, and simply, too.

But the main intent, it seems, is to lay a foundation for serious thought and study on numbers. In this the author has been successful, indeed. Unlike the writer of a text, he has taken time to write out sequences of numbers for his discussion so that the lay reader or curious young student can watch the development of the theory. This approach suggests a method of investigation to the nonprofessional. Many things are proved, but the author leaves much that is not. Still, the numerous specific examples suggest proofs which a student of some maturity can devise, or find.

A good short treatment of the positional representation is followed by two chapters on arithmetic series, which together with the chapter on primes would alone justify the book for high school students. There is a fine discussion on perfect numbers which includes the fact that the known perfect numbers are triangular numbers. The chapter on irrational numbers develops into a gratifying elementary discussion of continued fractions. There are interesting appendices, the last being an introduction of the geometric series as a tool of additive number theory.

Perhaps the strongest contribution the book makes is to demonstrate the

fundamental and far-reaching character of the arithmetic series. One wishes that an equal treatment had been given the geometric series at, say, the expense of magic squares. Even so, this little book should be very stimulating to high school and elementary mathematics teachers who are trying to revitalize their students' interest in numbers.

ED WALTERS
Wm. Penn High School

An Introduction to Probability Theory and Its Applications, Vol. I, 2nd Ed. By William Feller. Wiley, New York, 1957. xv+461 pp. $10.75.

The first edition of this book (Wiley, 1950, reviewed in this MONTHLY, vol. 59, 1952, p. 265) is so well-known that the only pertinent topic for the present review is the extent of the changes.

Feller has long been interested in the probable vagaries of a single sequence of trials as contrasted with the mean behavior of an aggregate of sequences. In this connection he has inserted a new chapter (Chapter III) in which he gives a very elegant, yet elementary, derivation of some rather startling theorems on coin tossing.

The major change is that the theory of recurrent events has been pushed forward so as to permeate the entire book, whereas in the first edition serious consideration of this theory began in Chapter 12. The result of this revision is a remarkable improvement in organization.

The preface to the second edition mentions "space saved by streamlining," but it must be emphasized that the streamlining is in organization, not in exposition. Explanations and examples have been expanded so that what was already an outstanding bit of exposition has been noticeably improved. Type has been completely reset so that these improvements occur on nearly every page. Kudos to Wiley for agreeing to scrap so many costly plates! The result is well worth the increase in price.

One big disappointment to the reviewer: failure to distinguish between a function and one of its values makes the introductory discussion of random variables confusing. Instead of clearing this up, Feller has deleted the one sentence in the first edition that hinted at the explanation. Had he only chosen to expand this now-missing sentence to two pages in his usual expository style, it would have been a real service to the teaching of mathematics in general.

M. E. MUNROE
University of Illinois

BRIEF MENTION

A Freshman Honors Course in Calculus and Analytic Geometry. By Emil Artin. Committee on the Undergraduate Program, Mathematical Association of America, 1957. 126 pp. Free.

The Committee on the Undergraduate Program of the Mathematical Association of America has prepared this edition of Seligman's notes on Artin's

course which is taught to the upper ten or fifteen per cent of the freshmen students enrolled in mathematics at Princeton University. It should certainly be on the active bookshelf of every mathematician interested in teaching gifted students. The Committee on the Undergraduate Program deserves the sincere thanks of mathematicians everywhere for making these notes available without charge. Copies may be obtained by writing directly to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Logical Design of Digital Computers. By Montgomery Phister, Jr. Wiley, New York, 1958. xvi+408 pp. $10.50.

This is not a mathematical book even though it contains two chapters on Boolean algebra, including the use of Veitch diagrams to simplify Boolean polynomials. It provides an excellent illustration of the extreme importance of modern abstract algebra in the engineering worlds of today and tomorrow.

Mechanical Resolution of Linguistic Problems. By Andrew D. Booth, L. Brandwood, and J. P. Cleave. Academic Press, New York, 1958. vii+306 pp. $9.80.

The results of the Birkbeck College Computational Laboratory on the application of digital computers to language translation problems are presented without reference to detailed programming. This book belongs in the library of every computing center as well as in departments of modern language.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor F. L. Wolf, Carleton College, represented the Association at the Observance of the Centennial of the Founding of Shattuck School at Faribault, Minnesota, on Friday, June 6, 1958.

Professor R. E. Wheeler, Head of the Department of Mathematics of Howard College, represented the Association at the inauguration of Henry King Stanford as President of Birmingham-Southern College on Friday, April 11, 1958.

Professor Morris Kline, New York University, has received a Fulbright award to lecture at Technische Hochschule in Aachen, Germany, during the academic year 1958–59.

Alabama Polytechnic Institute: Mr. P. W. Lindsey, Jr., has been promoted to Assistant Professor; Mr. S. M. Lukawecki has been appointed Instructor.

University of Detroit: Assistant Professor B. V. Ritchie has been promoted to Associate Professor; Mrs. Natalie Frazis, Mr. J. F. Lanahan, Mr. M. A. Laframboise, Mr. J. A. Mansour, and Mr. J. G. Sowul, have been promoted to Assistant Professors; Professor L. E. Mehlenbacher was a Visiting Professor and Assistant Director of the Summer Institute for High School Teachers of Science and Mathematics at the University of North Dakota during the summer of 1958.

Emory University: Associate Professor Trevor Evans has been promoted to Professor; Assistant Professor Henry Sharp, Jr., has been promoted to Associate Professor; Mr. B. K. Youse has been promoted to Assistant Professor; Mr. G. R. Flowers and Mrs. Peggie W. Wiegand have been appointed Instructors; Mr. F. L. Hardy begins a leave of absence in September in order to continue his graduate training.

University of Hawaii: Associate Professor S. B. Townes has been promoted to Professor; Assistant Professor Paola Comba has been promoted to Associate Professor; Captain J. H. Spiller, USN (retired), and Mr. I. C. Young, University of Colorado, have been appointed Assistant Professors.

Hobart and William Smith Colleges: Associate Professor R. L. Beinert has been promoted to Professor; Assistant Professor Abigail M. Mosey has been promoted to Associate Professor.

Illinois Institute of Technology: Associate Professor L. R. Wilcox has been promoted to Professor; Assistant Professors Pasquale Porcelli and R. J. Silverman have been promoted to Associate Professors; Dr. M. A. McKiernan and Dr. H. L. Pearson have been promoted to Assistant Professors.

Institute for Advanced Study, 1958–59: Dr. E. H. Batho, on leave from the University of Rochester, NSF grant; Associate Professor Alice T. Schafer, on leave from Connecticut College, NSF Science Faculty fellowship; Professor R. D. Schafer, on leave from the University of Connecticut, NSF Senior Postdoctoral fellowship.

Massachusetts Institute of Technology: Professor N. C. Ankeny was awarded a Guggenheim fellowship and will be on leave 1958–59 at Cambridge University; Professor G. W. Whitehead will be on leave as a visiting professor at Princeton University; Assistant Professor I. M. Singer has been promoted to Associate Professor; Assistant Professor J. K. Moser, New York University, visiting assistant professor during the current year, has been appointed Associate Professor; Dr. G. E. Backus, and Dr. F. P. Peterson, Princeton University, have been appointed Assistant Professors; Dr. A. P. Mattuck, Lecturer, who is spending the current semester in England, has been promoted to Assistant Professor; Doctors R. M. Peterson, New York University, E. R. Rodemich, Stanford University, and J. W. Smith, Columbia University, have been appointed C. L. E. Moore Instructors; Dr. A. W. Adler, Princeton University, has been appointed Lecturer; Dr. Shoshichi Kobayashi, Institute for Advanced Study, has been appointed Research Associate; Dr. James Yeh, University of Minnesota, has been appointed Instructor; Dr. R. J. Gribben, University of Manchester, England, and Dr. Azriel Levy, Hebrew University, have been awarded Sloan Foreign Post-doctoral fellowships in the School for Advanced Study at M.I.T.

St. Louis University: Associate Professor John Elder has been promoted to Professor.

University of Wisconsin, Mathematics Research Center, U. S. Army: Professor E. H. Zarantonello has returned to his post at the University of Cuyo, Mendoza, Argentina; Professor Zdenek Kopal is on leave, and has resumed his post at the University of Manchester, England; Professors H. F. Weinberger, University of Maryland, H. M. Schaerf, Washington University, and Fritz Oberhettinger, American University and National Bureau of Standards, are visiting members on leave from their Universities.

Dr. A. G. Anderson, General Tire & Rubber Co., has been appointed Professor and Head of Department, Western Kentucky State College.

Associate Professor D. F. Atkins, University of Richmond, has been appointed Assistant Professor at Eastern Illinois University.

Dr. J. L. Bailey, Michigan State University, has been appointed Assistant Professor at Case Institute of Technology.

Dr. Jeremiah Certaine, Howard University, has accepted the position of Manager, Department of Mathematics, Nuclear Development Corporation of America, White Plains, New York.

Mr. Willard Draisin, Lincoln Laboratories, has been appointed Instructor at Connecticut College.

Professor M. M. Day has been appointed Head, Department of Mathematics, at the University of Illinois, to succeed Professor S. S. Cairns.

Mr. E. S. Eby, University of Illinois, has accepted a position as Mathematician with U. S. N. Underwater Sound Laboratory, New London, Connecticut.

Mrs. Pauline P. Edwards, Los Alamos Scientific Laboratory, is now a Mathematician at the Aeronautical Chart & Information Center, St. Louis, Missouri.

Mr. J. R. Eno, Jr., University of Idaho, has been appointed Instructor at Whittier College.

Dr. R. E. Fagen, Bell Telephone Laboratories, has accepted a position as a member of the technical staff at Hughes Aircraft Company, Culver City, California.

Dr. H. H. Goldstine, Institute for Advanced Study, has been appointed to the staff of the I. B. M. Research Center, Yorktown, New York.

Mr. R. M. Gordon, Electro Data Division, Burroughs Corporation, Pasadena, California, has been promoted to Manager, Publications and Training.

Mr. R. J. Graham, Eglin AFB, Florida, is now a Mathematician with ABMA, Redstone Arsenal, Huntsville, Alabama.

Assistant Professor Bernard Greenspan, Drew University, has been promoted to Associate Professor. As a recipient of a National Science Foundation Faculty fellowship, he will be on leave at the University of California, Berkeley, 1958–59.

Mr. J. W. Haake, Armour Research Foundation, is now a Research Engineer with Convair, San Diego, California.

Mr. D. I. Hammer, Montclair State Teachers College, has been appointed Assistant Professor at Adelphi College.

Miss Virginia S. Hanly, Ohio State University, is now a Research Engineer at North American Aviation, Columbus, Ohio.

Professor F. F. Helton, Central College, Fayette, Missouri, has been appointed Chairman of the Division of Mathematics and Natural Sciences.

Dr. H. G. Hertz, U. S. Naval Observatory, is now with the Army Map Service, Washington, D. C.

Dr. T. P. Higgins, Boeing Airplane Company, is now a Senior Scientist with Dalmo-Victor, Belmont, California.

Dr. John Hilzman, Oregon State College, has been appointed Assistant Professor at Harpur College.

Professor J. H. Hodges, University of Buffalo, has accepted a position as Mathematician with Cornell Aeronautical Laboratories, Buffalo, New York.

Mr. Edgar Karst, International Business Machines Corporation, Endicott, N. Y., has been appointed Assistant Professor at Brigham Young University.

Dr. L. D. Kovach has been appointed Professor and acting head of the Department of Mathematics and Physics at George Pepperdine College. He will retain his position as design specialist at the Douglas Aircraft Company, El Segundo, California.

Dr. Joseph Lehner, Visiting Professor at Michigan State University, has been appointed Professor.

Dr. M. E. Levenson, Brooklyn College, has been promoted to Assistant Professor.

Dr. Eugene Levin, Ramo-Wooldridge Corporation, is now a Mathematician with the RAND Corporation, Santa Monica, California.

Professor G. G. Lorentz, Wayne State University, was Visiting Professor at the

University of Tübingen, Germany, during the second part of the Summer semester of 1958.

Mr. F. L. Lynch, Jr., Madison High School, Madison, New Jersey, is now Associate Professor at Seton Hall University.

Mrs. Matilde C. Macagno, formerly Professor on leave, University of Cuyo, San Juan, Argentina, is now Research Associate, Iowa Institute for Hydraulic Research, State University of Iowa.

Professor K. O. May, Carleton College, Northfield, Minnesota, will be on sabbatical leave in western Europe during the first semester of the academic year 1958–59.

Mr. R. W. McChesney, University of Rochester, has been appointed Instructor at Albion College, Michigan.

Mr. M. S. Moheban, Montana State University, is now a teacher at Park Senior High School, St. Paul, Minnesota.

Mr. M. G. Ossesia, University of Pittsburgh, has been appointed Instructor at Duquesne University.

Mr. G. P. Paternoster, Roosevelt University, is now Structural Detailer and Checker with Johnson & Johnson Engineers and Architects, Chicago, Illinois.

Dr. Sara L. Ripy, Vassar College, has been appointed Assistant Professor at Agnes Scott College.

Mr. Jack Roseman, D. I. C. Staff, Massachusetts Institute of Technology, has been appointed Instructor at the University of Massachusetts.

Mr. S. J. Scott, Vitro Corporation of America, is now Operations Analyst with North American Air Defense Command, Colorado Springs, Colorado.

Mr. R. T. Seeley, Massachusetts Institute of Technology, has been appointed Instructor at Harvey Mudd College.

Assistant Professor W. A. Small, Grinnell College, has been appointed Chairman of the Department of Mathematics. He was also Visiting Professor for the 1958 summer session at Nebraska State Teachers College, Chadron.

Mr. Irwin Stoner, Cornell Aeronautical Laboratories, is now Assistant Section Head, Raytheon Mfg. Co., Bedford, Massachusetts.

Professor D. D. Strebe, Teachers College at Oswego, State University of New York, has been appointed Associate Professor at the University of South Carolina.

Dr. Walter Bartky, University of Chicago, died March 19, 1958. He had been a member of the Association for thirty-one years.

Professor Daniel Block, Yeshiva University, died February 28, 1958. He had been a member of the Association for ten years.

Mr. W. M. Bullit, Louisville, Kentucky, died on October 3, 1957. He had been a member of the Association for thirty-two years.

Associate Professor Emeritus C. F. Lewis, Kansas State College, died on April 4, 1958. He had been a member of the Association for twenty-six years prior to his retirement.

Mr. P. W. A. Raine, Newport News High School, died October 14, 1957. He had been a member of the Association for eighteen years.

Professor Emeritus G. E. Ramsdell, Bates College, died February 2, 1958. He was a charter member of the Association.

Professor S. J. Smith, State Teachers College, Lock Haven, Pennsylvania, died January 21, 1958. He had been a member of the Association for twenty-seven years.

Dr. Marion B. White of Pasadena, California, died January 30, 1958. She was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1958 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	R. G. Smith, Kansas State College, Pittsburg
Missouri	W. R. Utz, Jr., University of Missouri
New Jersey	William Feller, Princeton University
Northeastern	F. M. Stewart, Brown University
Ohio	G. M. Merriman, University of Cincinnati
Pacific Northwest	A. T. Lonseth, Oregon State College
Southeastern	G. B. Huff, University of Georgia
Southwestern	Charles Wexler, Arizona State College at Tempe
Upper New York State	H. S. M. Coxeter, University of Toronto

The highest percentage of votes cast was in the Kansas Section, where votes were received from more than 52% of the membership.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 198 persons have been elected to membership by the Board of Governors on applications duly certified.

ROBERT D. ADAMS, B.S. (Northwestern) Teaching Asst., University of Minnesota.

JOHN J. AEBERLY, C. E. (Illinois Tech.) Chief, Bureau of Heating, Ventilation and Industrial Sanitation, Chicago, Illinois.

WALTER A. ALBRECHT, JR., Ph.D. (Ohio S. U.) Asso. Professor, Long Beach State College, California.

GARY E. ANDERSON, Student, Abilene Christian College.

MARYSE BADER, B.A. (California) Instr., Seattle University; Student, University of Washington.

MERVIN R. BARNES, JR., Student, University of California, Riverside.

MARGARET M. BASKERVILL, Ph.D. (Alabama Poly.) Professor, Shorter College.

CHARLES BATCHLOR, M.S. (Howard) Mathematician, U. S. Army Map Service, Washington, D. C.

JOHN E. BEAM, Student, University of Kansas.

ROBERT C. BELSCAMPER, M.S. (Iowa S.C.) Mathematician, Remington Rand Univac, St. Paul, Minnesota.

PHILLIP R. BENDER, B.S. (Purdue) Asst. Professor, Milwaukee School of Engineering, Wisconsin.

WILLIAM H. BENSON, M.S. (M.I.T.) Private, United States Army, Ft. Lee, Virginia.

PAULINE J. BIGGS, Student, University of Oklahoma.

ROBERT L. BLEFKO, M.A. (Penn. S.U.) Instr., Pennsylvania State University.

WILLIAM F. BLOSE, Student, Oklahoma State University.

SAMUEL B. BLUMERT, Student, Cooper Union.

JOHN M. BOSSERT, Student, University of California, Los Angeles.

GARY T. BOSWELL, Student, Texas Christian University.

GERALD A. BOTTORFF, Student, Seneca High School, Louisville, Kentucky.

BEN B. BOWEN, A.B. (U.C.L.A.) Teacher, Vallejo Junior College, California.

JOAN E. BRENIZER, M.A. (Texas) Instr., Lamar State College of Technology.

MRS. MARY L. BRIGHAM, M.A. (Baylor) Instr., Baylor University.

ARCHIE D. BROCK, B.S. (Northwestern S.C., Oklahoma) Grad. Student, University of Oklahoma.

ROBERT R. BROWN, M.A. (Southern California) Manager, Special Projects, I.B.M. Beverly Hills, California.

WILLIAM J. CAHILL, M.A. (Harvard) Senior Engineer, Boeing Airplane Co., Patrick A.F.B., Florida.

MELVAN D. CARTER, Student, University of Oklahoma.

NED Y. H. CHANG, Student, Washington & Jefferson College.

BRUCE L. CHILTON, Student, University of Buffalo.

PETER J. CHRISLER, M.A. (California) Teacher, Sacramento Junior College.

LESLIE C. COLLINS, A.F.R. Aero S. (Royal Aero. Soc.) Design Engineer, Sikorsky Aircraft, Bridgeport, Connecticut.

PETER S. COOK, B.S. (Nebraska S.T.C., Chadron) Asst. Professor, Scottsbluff College.

RICHARD W. COTTLE, B.A. (Harvard) Teach. Fellow, Harvard.

JOHN F. CUNNINGHAM, Student, St. Johns College, Jamaica, New York.

ALLEN S. DAVIS, Student, University of Oklahoma.

NATHAN L. DAVIS, B.S. (Ohio) Instr., Lancaster High School, Lancaster, Ohio.

LAURENCE A. DRYDEN, M.S. (Fort Hays Kansas S.C.) Instr., Fort Hays State College.

JOHN R. DURBIN, B.A. (Wichita) Teach. Fellow, University of Wichita.

EDWARD G. ELDER, Student, University of Oklahoma.

EUGENE FAMOLARI, JR., Student, Massachusetts Institute of Technology.

LEONARD FELDMAN, Ed.D. (California, Berkeley) Asst. Professor, San Jose State College.

MRS. E. MURIEL FERGUSON, Ph.D. (Washington, St. Louis) Asst. Professor, Washington University; Senior Research Asst., Social Science Institute, St. Louis.

ALVARO FERLINE, M.S. (Chicago) Asst. Professor, New Haven State Teachers College.

REV. JAMES J. FISCHER, S.J., M.S. (Fordham) Instr., Fordham University.

HENRY E. FLEMING, Student, Harpur College.

PHILIP FRANDINA, Student, University of Buffalo.

STUART M. FRIEDMAN, M.A. (Columbia) Chairman of Department, Los Angeles Harbor Junior College, Wilmington, California.

ROBERT H. FRYLING, M.S. (Pittsburgh) Asst. Professor, Gettysburg College.

JOHN N. FUJII, M.A. (Columbia, T.C.) Instr., Oakland Junior College (Merritt).

CHARLIE W. GAMMILL, B.S. (Delta S.C.) Acting Instr., Mississippi State College.

EDITH T. GARDNER, B.S. (Ft. Valley S.C.) Grad. Student, University of Chicago.

FRANK C. GERMAN, M.A. (Illinois) Professor, Kansas State Teachers College, Pittsburg.

RICHARD R. GERO, Student, Hofstra College.

NANCY L. GETCHELL, Student, University of Maine.

QAMARUDDIN K. GHORI, M.S. (Pakistan) Teaching Asst., University of British Columbia.

SCOTT K. GIBSON, M.S. (M.I.T.) Instr., Florida Southern College.

CHARLES H. GLADMAN, M.A. (Ohio S.U.) Asso. Professor, Texas Western College.

MICHAEL GOLDSTEIN, Student, Polytechnic Institute of Brooklyn.

PAUL M. GRABARKEWITZ, M.S. (N. Dakota S.C.) Instr., Bemidji State College.

MRS. MILDRED L. GROSS, B.S. (Nebraska S.T.C., Kearney) Grad. Asst., University of Nebraska.

STANLEY J. GUBERUD, B.S. (Wisconsin S.C., Eau Claire) Asst. Professor, Milwaukee School of Engineering.

IRVING HABER, Student, Cooper Union.

HWA S. HAHN, M.S. (Oregon) Grad. Student, University of Illinois.

CHARLES A. HALIJAK, Ph.D. (Wisconsin) Asso. Professor, Kansas State College, Manhattan.

ROBERT B. HALL, B.S. (Arizona S.C.) Teaching Asst., Arizona State College.

AUGUST J. HALTMAIER, M.A. (Boston) Teacher, Oceanside High School, New York.

LT. NEIL K. HANSEN, B.A. (Washington) Instr., U. S. Naval Academy, Annapolis.

DELBERT R. HARPER, B.S. (Tulsa) Research Physicist, Carter Oil Co., Tulsa, Oklahoma.

WILLIAM E. HARTNETT, Ph.D. (Kansas) Asst. Professor, College of the Holy Cross.

HARRY S. HAYASHI, M.A. (Southern California) Instr., Los Angeles City College.

D. FAYE HENDRIX, M.A. (Geo. Peabody) Instr., Lamar State College.

DONALD G. HESS, M.A. (Long Beach S.C.) Instr., Whittier High School, La Habra, California.

ROBERT HEYNEMAN, Ph.D. (California, Berkeley) Instr., Cornell University.

ROY H. HOLMGREN, M.A. (California) Instr., College of San Mateo.

ALBERT JAQUA, Ph.D. (U.C.L.A.) Los Angeles, California.

HELENE S. JOHNSON, Student, Texas Christian University.

THOMAS J. JOHNSON, III, Student, University of Oklahoma.

DAVID W. JONAH, M.S. (Purdue) Grad. Asst., Brown University.

WILLIAM B. JONES, M.A. (Vanderbilt) Grad. Fellow, Vanderbilt University.

STANISLAUS C. KARPACZ, Student, St. John's University, Jamaica, New York.

ALAN M. KATCHER, Student, Cooper Union.

ROBERT KATZ, B.S. (Chicago) Instr., Tufts University.

LEE H. KENNEDY, Student, Texas Christian University.

JOHN V. KENNY, B.C.E. (Glasgow) Senior Engineer, New Jersey State Highway Dept., Rockaway, New Jersey.

KENNETH P. KIDD, Ph.D. (Geo. Peabody) Professor, University of Florida.

CHARLES H. KING, Student, San Jose State College.

KERRY W. KING, Student, Abilene Christian College.

TRUMAN S. KLEIN, M.A. (Maryland) Supervisor of Secondary Education, Board of Education, Upper Marlboro, Maryland.

LADIS D. KOVACH, Ph.D. (Purdue) Design Specialist, Douglas Aircraft Co., El Segundo, California.

JESSE D. KREHBIEL, M.S. (Iowa S.C.) Instr., University of Wichita.

ARLENE S. KROCHMAL, A.A. (Geo. Washington) Student, George Washington University.

ORVILLE C. KRUSCHWITZ, M.A. (Geo. Peabody) Instr., Texas Western College.

WILLIAM R. LACOUR, Student, McNeese State College.

MRS. BETTY P. LAMKIN, Student, University of Oklahoma.

NEAL F. LANE, Student, University of Oklahoma.

PHILIP W. LATIMER, M.S. (N. Texas S.) Asso. Professor, Lamar State College.

ROBERT W. LAWRENCE, Student, University of Oklahoma.

SOLOMON LEADER, Ph.D. (Princeton) Asst. Professor, Rutgers University.

GEORGE F. LEHMAN, B.A. (Minnesota) Employment Interviewer, Iowa State Employment Service, Davenport, Iowa.

JAMES B. LEWIS, M.A. (Mississippi) Asst. Prof., McNeese State College.

DON R. LICHTENBERG, B.A. (Iowa S.T.C.) Grad. Student, University of Wisconsin.

WILLIAM M. LINDSTROM, B.S. (Kansas) Grad. Student, University of Kansas.

M. ALBERT LINTON, JR., A.M.T. (Harvard) Head of Department, Wm. Penn Charter School, Philadelphia, Pennsylvania.

BIRGER LÖVGREN, Fil. Mag. (Stockholm) Grad. Student, Cornell University.

EUGENE T. LUCAS, M.A. (Denver) Asst. Professor, Oklahoma Baptist University.

EUGENE M. LUKS, Student, The College of the City of New York.

JOHN J. LUND, Ph.D. (Chicago) Head of Department, Lassen Junior College.

ROBERT J. LYON, Student, Colorado State College.

KENNETH A. MACDONALD, B.A. (Arizona) Grad. Asst., University of Vermont.

MRS. VIVIENNE M. MAYES, M.A. (Fisk) Instr., Paul Quinn College.

GEORGE S. MCCARTY, JR., B.A. (California, Berkeley) Grad. Student, University of California, Los Angeles.

ROBERT D. MCCOY, Student, University of Oklahoma.

JOSEPH C. MCCULLY, Ph.D. (Michigan) Asst. Professor, Western Michigan University.

ROSS M. MCDONALD, B.S. (Tulsa) Design Engineer, Douglas Aircraft Co., Tulsa, Oklahoma.

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VALERIE MIKÉ, B.A. (Manhattanville) Tech. Staff Asso., Bell Telephone Labs., New York.

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HARLAN D. MILLS, Ph.D. (Iowa State) Research Asso., Princeton University.

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CAPT. DONALD O. NORRIS, B.S. (Chicago) Mathematician and Pilot, USAF, Wright-Patterson AFB, Dayton, Ohio.

ROBERT F. NORRIS, Student, Arizona State College.

NORBERT L. J. OLDANI, Student, University of Detroit.

MRS. KERAMANEH P. OSCHWALD, M.A. (Vanderbilt-Peabody) Instr., Northwest Missouri State College.

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THOMAS C. PATTEN, Student, University of Oklahoma.

DAVID M. PERKINS, Student, Pennsylvania State University.

RICHARD E. PETERSON, M.S. (Purdue) Instr., University of Dayton.

PAUL E. PFEIFFER, Ph.D. (Rice Institute) Asso. Professor, The Rice Institute.

ROBERT PHILLIPS, Student, Wm. L. Dickinson High School, Jersey City, New Jersey.

MRS. BERNICE V. H. POLLYEA, B.S. (Chicago) Teacher, Fenger High School, Chicago, Illinois.

CAPT. WILLIAM T. POWELL, JR. (retired) B.S. (U. S. Naval Academy) United States Navy.

CLARENCE E. PRINCE, JR., B.S. (Southern Methodist) Instr., Taejon Presbyterian College, Taejon, Korea.

MRS. CANDIDA K. REES, M.S. (Virginia Poly. Inst.) Instr., University of Illinois.

CHARLES A. REIHER, Student, St. John's University, Jamaica, New York.

JIMMY M. RICE, M.S. (Eastern New Mexico) Instr., Fort Hays Kansas State College.

MELVIN B. RICHINS, B.S. (Arizona S.C.) Asst. Instr., Arizona State College.

ARTHUR M. RIEHL, B.S. (Indiana) Grad. Asst., Purdue University.

LELAND F. RITCEY, Ph.D. (Chicago) Chairman of Department, University of Manitoba.

ROBERT A. ROTHSTEIN, Student, Massachusetts Institute of Technology.

ANDREW W. ROW, M.A. (Michigan) Instr., University of Florida.

WILLIAM B. RUNDBERG, Student, San Jose State College.

DONALD E. RYERSON, B.Com. (Toronto) Teacher, Toronto Board of Education.

DON E. RYOTI, Student, Northwestern University.

HANS SEGAN, Ph.D. (Vienna) Asso. Professor, University of Idaho.

THOMAS W. SAMPSON, M.A. (Western Ontario) Head of Department, Blakelock High School, Oakville, Ontario.

ROBERT L. SANDERS, Student, St. Marys University, Texas.

JULIUS SASLOW, Student, Cooper Union.

RICHARD P. SAVAGE, M.S. (Oklahoma S.U.) Instr., University of Tennessee.

FRED W. SCHLECHT, Student, Rutgers University.

BEAL A. SCHREINER, Clerk-Dispatcher, United States Government, Marine City, Michigan.

CALVIN F. SCHWARTZKOPF, Student, University of Wichita.

GEORGE E. SCHWEIGERT, Ph.D. (Johns Hopkins) Asso. Professor, University of Pennsylvania.

JAMES B. SERRIN, Ph.D. (Indiana) Asso. Professor, University of Minnesota.

JAMES F. SHEPPARD, Student, University of Oklahoma.

RICHARD L. SHOBE, B.S. (Alabama Poly. Inst.) Teach. Fellow, Alabama Polytechnic Institute.

JOHN W. SHUE, B.S. (Colorado) Instr., University of Colorado.

DONALD M. SILBERGER, B.S. (Harvard) Teacher, Shaw High School, Cleveland, Ohio.

JACK H. SILVER, Student, Montana State University.

RODNEY W. SIMISTER, B.S. (Queens) Mathematician, Bendix Aviation, Teterboro, New Jersey.

SISTER M. PETER (STENGL) B.A. (Holy Family Coll.) Holy Family College, Wisconsin.

LAWRENCE B. SKLAR, M.S. (Pennsylvania S.U.) Grad. Student, Rutgers University.

MRS. JANET SMASHEY, Student, Butler University.

KENNETH V. SMITH, Student, University of Texas.

SAM SPEED, JR., Student, Maryville College, Tennessee.

ROBERT L. STANLEY, Ph.D. (Harvard) Asst. Professor, Washington State College.

DALE W. STEIN, B.S. (Carnegie Tech.) Senior Scientist, Litton Industries, Beverley Hills, California.

DAVID J. STEINDLER, Student, Wesleyan University.

BENJAMIN STERN, M.S. (California Inst. Tech.) Instr., University of Omaha.

RICHARD H. STEVENS, M.A. (Cincinnati) Asst. Prof., University of Cincinnati.

RICHARD R. SWIFT, M.S. (Marquette) Racine, Wisconsin.

KENNETH W. THIESSEN, M.Ed. (Western Washington College) Chairman of Department, Skagit Valley College.

ROBERT J. THOMAS, M.S. (Indiana) Grad. Asst., University of Illinois.

GEORGE TIPTON, SR., M.S. (Colorado) Head of Department, Henderson County Junior College, La Rue, Texas.

LINCOLN H. TURNER, Ph.D. (Purdue) Asst. Professor, Purdue University.

BILL L. TURNEY, D.Ed. (Houston) Asst. Professor, McNeese State College.

FREDERICK R. URBANUS, M.S. (DePaul) Scientist, Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

ROBERT E. WALDEN, Student, Hendric College.

MRS. LINA R. WALTER, M.A. (Columbia) Asso. Prof., New Jersey State Teachers College, Paterson.

CHUNG-LIE WANG, B.S. (Ordnance Eng. College, Taiwan) Grad. Student, University of South Carolina.

PAUL M. WEICHSEL, M.S. (New York) Teach. Asst., California Institute of Technology.

DAN B. WELLS, B.S. (Western Carolina) Instr., University of Kentucky.

DAVID B. WELLS, B.S. (Nebraska) Instr., University of Nebraska.

BRO. THOMAS C. WESSELKAMPER, B.S. (Dayton) Instr., Chaminade High School, Dayton, Ohio.

MRS. LYLE M. WHEELER, M.A. (Oregon S.C.) Head of Department, Astoria High School, Oregon.

ROBERT I. WHITE, B.A. (Pennsylvania) Chairman of Department, Cheltenham High School, Philadelphia, Pennsylvania.

JOHN E. WHITESITT, Ph.D. (Illinois) Asso. Professor, Montana State College.

JAMES L. WIDMAIER, Student, Otterbein College.

DOUGLAS W. WILLETT, Student, South Dakota School of Mines and Technology.

RALPH C. WILLIAMS, M.S. (California Inst. Tech.) Teacher, Pasadena City College.

MRS. VELMA M. WILLIAMS, M.S. (DePaul) Instr., Wiley College.

WILLIAM S. WORLEY, JR., Student, Phillips University.

ROBT. H. WORONUK, Student, University of British Columbia.

NSF GRANTS TO THE ASSOCIATION

The Association has recently received the following grants from the National Science Foundation.

A grant of $7,500 has been received for a conference to review the program of the Association and to formulate a plan of action. This conference, which was held in Washington, D. C. on May 16–18, will be reported in more detail in the October issue of the MONTHLY.

A grant of $8,000 was awarded for the support of a preliminary study of nonteaching mathematical employment. An advisory committee for this survey has been appointed, consisting of Morris Ostrofsky (Chairman), Wallace Givens (Acting Chairman), Paul Armer, T. E. Caywood, Churchill Eisenhart, Z. I. Mosesson, G. B. Thomas, Jr.

A grant of $47,700 has been received for the support of a pilot program of visiting lectureships to secondary schools for a two-year period. To administer this program, the following Committee on Secondary School Lecturers has been appointed: J. R. Mayor, Chairman (1958–1960), Roy Dubisch (1958–1960), W. E. Ferguson (1958–1961), B. W. Jones (1958–1959), Mrs. Marie S. Wilcox (1958–1961).

A grant of $49,000 has been received for the support of a program of production of films for improving collegiate mathematics.

THE NINETEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The nineteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, November 22, 1958. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly-enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed out early in September to the regular mailing list. If an application blank is not received by September 20, you may secure one by writing the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 1, 1958. For further details of the examination and the list of prizes (including the $2500 scholarship at Harvard) see the announcement which will be mailed out along with the application blank.

Reports of the eighteen previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947; December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; and this issue, pp. 515–516.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 22, 1958 at the University of Michigan, Ann Arbor, Michigan in conjunction with the meeting of the Michigan Academy of Science, Arts and Letters. Professor A. E. Lampen of Hope College presided at both the morning and afternoon meetings and at the luncheon business meeting. Total attendance was 108, including 70 members of the Association.

The nominating committee, consisting of Professors P. S. Dwyer, Chairman, H. D. Larson and S. D. Conte, proposed for Chairman, Professor G. Y. Rainich, University of Michigan; Vice-Chairman, Professor W. D. Baten, Michigan State University; and Secretary-Treasurer, Professor F. A. Beeler, Western Michigan University. The slate was elected unanimously.

The Michigan Mathematics Prize Competition Committee, consisting of Professors R. H. Oehmke, A. J. Lohwater, A. W. Jacobson and F. L. Celauro gave a report of its activities during the past year. The Committee, with the approval of the Executive Committee, decided to go ahead with plans for a competitive examination for High School students this Spring. This examination will be given March 27, 1958 in 315 high schools to 6100 students. The Committee felt that it would be better to write its own examination and also to score the papers. The examination, based on the subject matter of the four-year high school mathematics curriculum, is in two parts. The first part, a multiple-choice type examination, is designed to test the general mathematical back ground of the contestant. The second part is of a written character and is designed to measure the

mathematical maturity of the contestant. Such factors as originality and clarity of expression will be weighed. Medals, certificates, prizes and scholarships will be awarded to the highest scorers. These are given by the sponsoring organizations; colleges, universities and industries of the area. The Section passed a resolution to sponsor such an examination.

Professor Leo Goldberg, Head of the Department of Astronomy at the University of Michigan, was the special guest speaker at the meeting. Speaking on the topic "Solar Physics," Professor Goldberg explained various physical phenomena associated with the sun and methods used in investigating these phenomena. His lecture was illustrated with slides. At the conclusion of the lecture a special motion picture film, prepared by a group of European astronomers, was shown.

The following papers were presented:

1. *A particular affinely-connected manifold*, by Dr. N. J. Hicks, University of Michigan, introduced by the Secretary.

The purpose of this note is to exhibit a simply-connected C^∞ manifold M on which is defined a complete affine connection satisfying the following property: for any integer $n \geqq 0$ there are two points p and q in M such that any broken geodesic from p to q must contain at least n breaks. The manifold M is not compact; however, for each n there is a compact simply-connected manifold for which one must allow at least n breaks in order to connect all pairs of points with broken geodesics.

2. *The solution of certain differential equations by Laplace transforms*, by Professor H. E. Stelson, Michigan State University.

The solution of linear differential equations with polynomial coefficients is considered. Properties of the differential operator $\overline{D}$, $[X^m D^m = \overline{D}(\overline{D}-1)(\overline{D}-2)\cdots(\overline{D}-m+1)]$ are used as an aid in the solution. Especially the relation $\phi(\overline{D})X^n = \phi(n)X^n$ is used. The reducibility of differential coefficients is facilitated by using the operator $\overline{D}$. The property of Laplace transforms, $L(\overline{D}^n y) = (-1)^n (d^n/dp^n)[p^n L(y)]$ is used in the solution.

3. *Stieltjes product integrals*, by Professor T. H. Hildebrandt, University of Michigan.

While the Stieltjes integral $\int_a^b f d\alpha$ for $f(x)$ continuous on (a, b) and $\alpha(x)$ of bounded variation exists as the limit of the sums $\sum_{i=1}^n f(x_i')[\alpha(x_i)-\alpha(x_{i-1})]$ as the maximum length $|x_i - x_{i-1}|$ of the intervals of the subdivision $a = x_0 < x_1 < \cdots < x_n = b$, with $x_{i-1} \leqq x_i' \leqq x_i$, approaches zero, this does not hold for the product $\prod_{i=1}^n \{1 + f(x_i')[\alpha(x_i)-\alpha(x_{i-1})]\}$ connected with the Stieltjes product integral $(P)\int_a^b [1 + f(x)d\alpha(x)]$ if $\alpha(x)$ is discontinuous. However, the convergence is valid in the sense of successive subdivisions. While $(P)\int_a^b [1 + f(x)dx]$ is equal to $\exp \int_a^b f(x)dx$ if $f(x)$ is Riemann integrable, and $(P)\int_a^b [1 + d\alpha(x)] = \exp [\alpha(b) - \alpha(a)]$ if $\alpha(x)$ is continuous and of bounded variation, terms which are not exponential in character appear if $\alpha(x)$ is discontinuous. Moreover, when $\alpha(x)$ is continuous $(P)\int_a^x [1 + d\alpha(x)]$ satisfies the integral equation analogous to a linear differential equation *viz.* $y(x) = \int_a^x [d\alpha(x) y(x) + 1]$, but this is no longer always true if $\alpha(x)$ is discontinuous. Similar remarks hold when $f(x)$ and $\alpha(x)$ are replaced by matrices with matrix multiplication.

4. *Stone-Weierstrass theorem and Komolgorov's consistency theorem of statistics*, by Professor Shu-Teh C. Moy, Wayne State University.

From the point of view that a probability measure defined for Borel subsets of a compact Hausdorff space Ω is a positive linear functional on the Banach space $C(\Omega)$ of continuous functions on Ω, the Komolgorov's consistency theorem of statistics is derived from the Stone-Weierstrass theorem.

5. *Parabolic analogues on theorems on harmonic functions*, by Dr. F. W. Gehring, University of Michigan, introduced by the Secretary.

Let $u = u(x, t)$ be a function defined in a domain D in the xt-plane. If u has continuous second partial derivatives and if $u_t = u_{xx}$, then u is in H over D. If $u = u_1 - u_2$ where u_1 and u_2 are non-

negative and in H, then u is in H^{Δ}. Functions in H have many properties in common with harmonic functions. Analogues of the maximum and of simple Phragmén-Lindelöf extensions are mentioned. When u is in H^{Δ} over the strip $0<t<c$, there is an important representation theorem due to Widder. Various forms of the Fatou theorem and its converse are discussed for such functions along with some uniqueness theorems and some parabolic analogues of the second theorem of Harnack and the Vitali convergence theorem.

6. *The effect of digital computers on university mathematical education*, by Professor J. W. Carr III, University of Michigan.

Digital computers are having a revolutionary impact on society. They will have a revolutionary impact on the teaching of mathematics. Since their use emphasizes the *generation of the algorithm* to solve, rather than the *actual solution* of the problem, much of the present rote solution of problems should be eliminated. Emphasis must be more on an abstract algebraic approach to teaching. The use of *command languages*, basic to computer use, should be considered, rather than the present descriptive languages. The actual use of computers in numerical and analytical (differentiation, integration, decision) problems, must be included in the standard texts.

7. *A computational algorithm for logical analysis*, by Mr. R. R. Korfhage, University of Michigan.

In order to make use of such logical methods as *reductio ad absurdum* an electronic computer must be able to derive the structure of any given logical expression. An algorithm is presented enabling the computer to do this for expressions written in the Polish prefix notation. The procedure requires only two scans of the expression. It can be easily adapted to other computational schemes ranging from arithmetic to mechanical translation procedures, and analogous algorithms can be written for systems using other than the Polish notation.

8. *Some electrical examples to illustrate Stokes' theorem*, by Professor W. P. Reid, Michigan State University

With the aid of Maxwell's equation for the curl of the magnetic field, H, one may use Stokes' theorem to determine H_θ around a straight wire carrying a steady current; the field due to steady currents flowing in various figures of revolution; the electric and magnetic fields inside a charging condenser; and the electric and magnetic field outside a charging sphere. Also, if one assumes that the electric field due to an oscillating dipole or to certain electro-magnetic waves traveling along circular wave guides is known, then H_θ may be calculated for these cases.

9. *Retention of mathematics by college freshmen*, by Professor F. L. Celauro, Central Michigan College.

Retention of knowledge and skills in trigonometry was investigated in relation to elapsed time, amount learned, intelligence and general mathematical proficiency. Students were tested upon commencing the study of trigonometry. After completing the course they were tested for achievement, and tested again some weeks later for delayed recall. Retention in trigonometry falls at a decelerated rate, 37 per cent retained after thirty weeks. Retention varies with different topics, identities most easily forgotten. Generalizations depend upon definition of retention employed (amount versus per cent retained). Retention is aided by injecting meaning and unification into the subject and viewing it in relation to other fields.

10. *Convergence in the mean of Taylor series*, by Dr. D. S. Greenstein, University of Michigan.

Given $f(x)$, a c^∞ function all of whose derivatives belong to $L^p(-\infty, \infty)$ for some fixed $p \geq 1$, the author considers representing the translates $f(x+h)$ by means of a "Taylor series in the mean." The class of functions so representable are generalizations to L^p of analytic function classes in L^2 studied by Paley and Wiener.

11. *A search for analogues of the Mathieu groups*, by Dr. E. T. Parker, University of Michigan and Mr. P. J. Nakolai, Ohio State University, presented by Dr. Parker.

The Mathieu groups of degrees 11, 12, 22, 23, and 24 are the only known simple finite groups contained in no known infinite system. Four of these a e the only known quadruply transitive permutation groups not symmetric or alternating. Degrees 11 and 23 are both of the form $p=2q+1$, with p and q primes. The analogues are simple transitive permutation groups of degree $p=2q+1$, of order $>p$ and $p!/2$. The authors first described pairs of generators of such groups, then wrote a program for UNIVAC Scientific Computer, Model 1103A. Groups of degree $p=2q+1$, $23<p\leqq 1823$, were examined exhaustively by the computer. No analogues exist in this range. Some six hours of machine time were required.

12. *The combination of $2\times n$ contingency tables*, by Dr. W. M. Kincaid, University of Michigan, introduced by the Secretary.

Let x and y be two experimental variables and let m trials, having outcomes "success" or "failure," be conducted under each of the kn conditions $x=x_i$, $y=y_j$, $i=1, \cdots, k$, $j=1, \cdots, n$; thus the results form a set of $2\times n$ contingency tables. The hypothesis that the probability of success is independent of y for each x may be tested by pooling the entries to form a single table and computing χ^2, but any dependence of the probability on x causes a spurious loss of significance. The purpose of the present paper is to show how this difficulty may be avoided.

(This work was partly supported by the Office of Naval Research.)

F. A. BEELER, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-fifth meeting of the Iowa Section of the Mathematical Association of America was held at Drake University, Des Moines, Iowa, April 18, 1958. Professor A. H. Blue, Chairman of the Section, presided. The total attendance was 71, including 30 members of the Association. Routine business was considered during the afternoon meeting. As a result of the work by the Committee on Contests appointed a year ago, it was voted that the Committee be commended for their work and that the officers of the Section be empowered to go ahead with high school contests in mathematics on a limited basis.

It was agreed that the Iowa Section would meet jointly with the State University of Iowa's Annual Conference of Teachers of Mathematics in October, 1958.

The following officers were elected: Chairman, Professor E. N. Oberg, State University of Iowa, Iowa City, Iowa; Vice-Chairman, Professor R. S. Jacobsen, Luther College, Decorah, Iowa; Secretary-Treasurer, Professor E. L. Canfield, DrakeUniversity, Des Moines, Iowa.

The following papers completed the program:

1. *Report of the committee on the problem of contests*, by Professor I. H. Brune, Iowa State Teachers College, Chairman.

The following points were made in the report of the Committee on Contests: (1) Mixed reac" tions were noted by high school teachers, some in favor, some opposed to contests. (2) Many teachers reported contests were not permitted. (3) Members of the advisory board of the Iowa Association of Mathematics Teachers seemed not to be in favor of the contests. (4) A poll of teachers in convention at Des Moines resulted in 44 votes in favor and 13 votes against. (5) While the Committee itself saw merit in a talent search it doubted that a contest should in any way pit school against school or teacher against teacher. Rather the contest should be a search for talent, not a device for rating teacher efficiency. (6) The Committee recommended that the Iowa Section of Mathematical Association of America try the contest for one year, taking advantage of the offer of the National Committee to conduct the examination but that time remaining to prepare for a contest for 1958 is doubtful; consequently, 1959 would be a better beginning date.

2. *Some curious results in distance geometry*, by Professor L. M. Blumenthal, University of Missouri. (By invitation.)

The results discussed concerned (1) non-congruent simplices with the same edges, (2) simplex-producing combinations of simplices, (3) a mapping of a bipunctured n-sphere onto an $(n-1)$-sphere, (4) an uncountable class of nonrectifiable arcs of Hilbert space, and (5) metric continua without small acute triangles.

3. *Grading systems*, by Professor Fred Robertson, Iowa State College.

The author compared several grading or testing systems, in industry and schools, with the one commonly considered as the grading system.

4. *A note on defining an extension of a probability measure on subsets of function space by applying one of J. L. Doob's theorems*, by Professor W. A. Small, Grinnell College.

Let W be the set of real-valued functions, $w(t)$, of the real variable t. An extension (W, F_2, P_2) of a Fundamental Borel Probability Field (W, F_0, P_0) was defined by J. L. Doob and S. Kakutani. It may happen that an adjunction extension (W, F_0', P_0') of (W, F_0, P_0) exists through adjoining a subset W' of W to F_0. By applying one of Doob's theorems, the condition on the outer P_2 measure $P_2^*(W')=1$ is seen to be necessary and sufficient for the existence of the corresponding adjunction extension (W, F_2', P_2') of (W, F_2, P_2). If (W, F_0', P_0') is measurable, then so is (W, F_2', P_2').

5. *An unusual method of teaching logarithms*, by Professor Fred Robertson, Iowa State College.

The author stressed the laws of operation needed for computation. The use of the tables is taught as an entirely different phase of the work. The tables of natural logarithms may be used first. Then a suggestion is made to change the tables of common logarithms. The terms characteristic and mantissa are not introduced.

6. *Uniform convergence of the second differences*, by Mr. U. R. Kodres, Iowa State College.

A sequence of theorems, whose proofs were sketched, was used to characterize the class of functions whose second differences converge to zero uniformly.

7. *Exceptional values of metric density*, by Mr. N. F. G. Martin, Iowa State College.

The usual definition of the metric density of a measurable set in E_1 at a point of E_1 is given. Then for a given real number λ, $0<\lambda<1$, a set is constructed whose density exists at 0 and has the value λ.

8. *Note on the classical canonical form of a matrix*, by Mr. J. C. Mathews, Iowa State College.

A proof of the existence and uniqueness of the classical canonical form of a matrix is accomplished without the intervention of invariant factors, elementary divisors, or modules. Using simple induction proofs a matrix A defined over an algebraically closed field is reduced to an intermediate canonical form B. At this point a final similarity transformation R is constructed such that $R^{-1}BR$ is classical. The proof of the uniqueness of the classical form of A is done by comparing ranks.

9. *Functions whose second difference goes to zero*, by Professor S. D. Nolte, Iowa State College.

If $f(x)$ is defined on an open interval I, the second difference is defined to be $|f(x+h)-2f(x)+f(x-h)|$. If a sequence $f_n(x)$, converges uniformly on I to $f(x)$ and if $f_n(x)$ is such that $\lim_{h\to 0}|f_n(x+h)+2f_n(x)-f_n(x-h)|=0$ for all h and all x in I, then $f(x)$ also has this property.

Definition. A function $f(x)$ is said to satisfy a second difference Lipschitz condition of order α on I if there exists an M and a $\delta>0$ such that $|f(x+h)-2f(x)+f(x-h)|<M|h|^{\alpha}$ for all x in I and all $|h|<\delta$. If $f(x)$ in this definition is continuous at one point in I, and if $\alpha\geqq 1$, then $f(x)$ is continuous at every point in I.

10. *Maxima of functions*, by Mr. J. D. Miller, Iowa State College.

By first defining what is meant by a real-valued function of a real variable taking on a proper relative maximum or a nonproper relative maximum at a point, it is shown that a function can possess at most a denumerable number of proper relative maxima. Furthermore, if a function takes on a relative maximum at every point of its domain of definition, then the range of the function is at most denumerable.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-third annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State Teachers College, Emporia, Kansas, on April 12, 1958, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 178 persons registered, including 52 members of the Association. Professor L. E. Laird, Chairman, presided at the sessions.

The following officers were elected for one year terms: Chairman, Professor P. S. Pretz, St. Benedict's College; Vice-Chairman, Professor J. D. Haggard, Kansas State Teachers College, Pittsburg; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State Teachers College, Pittsburg.

At the joint session, held in the morning, Dean Albert E. Meder, Jr., Rutgers University, delivered an address entitled "Modern Mathematics and Its Role in Secondary Education."

The following short papers were presented at the afternoon session:

1. *Statistics*, by Dr. Stanley Wearden, Kansas State College of Agriculture and Applied Science, introduced by the Secretary.

Rainfall information was obtained from the *United States Department of Commerce Weather Bureau Climatological Data*. Only data from the Hays, Kansas, reporting station was used. Median annual rainfall over the ninety years was 22.59 inches. Each year was classified as being above or below median rainfall, and the probability that the pluses and minuses occurred in an essentially random sequence was calculated. There were only thirty-six runs, or cycles, among the ninety years ($P=.034$). The correlations between monthly and annual rainfall were calculated. Ten correlations were significant ($P=.05$), and June rainfall explained over half the variability of annual rainfall.

2. *Braids*, by Professor J. C. Lillo, University of Kansas, introduced by the Secretary.

This paper consisted of expository comments on the theory of braids.

3. *What's wrong with mathematics*, by Professor C. B. Read, University of Wichita.

Without exhaustive or necessarily representative coverage, the paper presented some criticisms of mathematics which have appeared in professional periodicals. There was no attempt to give specific reference to any quotation, but it was suggested that listeners try to identify the person or group to which the point of view might be attributed. At the conclusion the fact was revealed that every quotation was written prior to 1932. The tentative conclusion was that, if, after twenty-five years, almost identical criticisms can be made, one wonders if mathematicians are making rapid progress towards revision and improvement.

4. *Calculus—a modern approach*, by Professor W. C. Janes, Kansas State College of Agriculture and Applied Science.

The idea of function as a consistent class of quantities with domain and range is developed early. A symbol for the identity function is helpful. Geometric considerations lead naturally to the fundamental theorem of the integral calculus. The derivative is defined as the limit of a difference quotient. Discussions of increments, infinitesimals, and differentials are almost nil, though in applications of integration the differential is retained as reminiscent of the type of product-sum whose limit is obtained. *Menger's Calculus—A Modern Approach* is an excellent book exemplifying the method.

5. *Are computing machines replacing mathematicians?* by Professor U. W. Hochstrasser, University of Kansas, introduced by the Secretary.

The capabilities and limitations of electronic computers are discussed and compared with the characteristics a mathematician is supposed to have. This comparison shows that the computer, at least in its present form, can not replace the mathematician and that its efficient and sensible use frequently depends essentially on the mathematical capabilities of its user. The widespread introduction of computers has accordingly not decreased the demand for mathematicians but caused a serious shortage of people with adequate mathematical training, a situation which can be only improved upon by offering a broader and better mathematical education at academic institutions.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section of the Mathematical Association of America was held on April 26, 1958 at the University of Kentucky, Lexington, Kentucky. Professor J. C. Eaves, University of Kentucky, presided at both the morning and afternoon sessions. There were 71 persons in attendance, including 49 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor W. J. Robinson, Centre College of Kentucky; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor Sallie E. Pence, University of Kentucky.

By invitation of the Committee, Professor Ernst Snapper of Miami University delivered an hour address at the afternoon session entitled, "Geometric Algebra." An abstract of this address follows:

The term "Geometric Algebra" was coined by Professor Emil Artin whose book by that title has recently appeared. One of the messages this book brings us is that the notion of a matrix, thought of as a rectangle of field elements, should be very much reduced in importance. Generally speaking, a matrix is an analytic representation of some geometric object, relative to a chosen coordinate system. If the same geometric object is represented by means of another coordinate system, we get another matrix. Consequently, if we treat a geometric object by means of matrices, we do not get an invariant theory. By now we have the experience that for the purposes of modern algebra invariant theories are the best. This change from "dealing with the geometric object through the intervention of a coordinate system" to "dealing with the geometric object directly, *i.e.*, doing geometric algebra," constitutes one of the major changes from the algebra of, say, 120 years ago to present-day algebra. As an example of geometric algebra, quadratic forms were discussed. A quadratic form was defined as a metric vectorspace, and the theory of metric vectorspaces was described through Witt's theorem.

The following papers were presented.

1. *The effects of prime characteristic in differential algebra*, by Professor Frank Levin, University of Kentucky.

In this talk the author cited various difficulties which arise in differential algebra when the groundfield has a characteristic $p \neq 0$.

2. *A coefficient problem for Laurent series*, by Miss Betty C. Detwiler, University of Kentucky.

In this paper the author studied analytic functions having a positive real part in an annulus. New proofs were obtained for well-known theorems due to Nehari and Robertson.

3. *Objectives of modern high school trigonometry*, by Professor Edith R. Schneckenburger, University of Buffalo.

Recommendations of the Commission on Mathematics were considered. The following objectives were suggested: mastery of subject matter needed for calculus and science courses, development of understanding of the nature of mathematics, appreciation of applications of trigonometry, improvement in the reading and writing of mathematics.

4. *Freshman mathematics at the University of Louisville*, by Professor W. L. Moore, University of Louisville.

A program was outlined with the objective of encouraging the more gifted students to take special courses in mathematics.

5. *The Mathematics of a wave tank*, by Professor W. J. Robinson, Centre College of Kentucky.

This paper presented the basic mathematics of a wave tank with a submerged transverse cylinder. Certain elementary properties were discussed, as well as one approximation to the true solution.

6. *Multiplication on the real line*, by Professor J. G. Horne, University of Kentucky.

All of the continuous associative multiplications of the real line which agree with the ordinary multiplication for the nonnegative reals are determined. In particular it is shown that Fawcett's characterization of ordinary multiplication on the closed-unit interval extends to a characterization of ordinary multiplication on the set of reals.

7. *On solving a certain nonlinear differential equation of 2nd order*, by Miss Dorothy I. Koehler, University of Kentucky.

The differential equation considered is taken from *On the solution of a differential equation with nonlinearity appearing in the second derivative of combined linear and cubic terms* by Chi-Neng Shen, Quart. Appl. Math., vol. 15, 1957, pp. 11–30. In particular we study the equation $d^2(y+\beta y^3)/dx^2+y=r$. The method of solution presented in this note varies from the method in the reference in that upon making the transformation $v^2=t$, the resulting equation is a first order linear differential equation. This is then solved by the usual methods.

8. *A network of checks for the multiplication of matrices*, by Professor J. C. Eaves, University of Kentucky.

In this paper the author indicated various checks that may be employed in the multiplication of matrices. This material is related to modern high speed computers.

V. F. COWLING, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Spring Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Randolph-Macon Woman's College, Lynchburg, Virginia, on April 26, 1958, with 70 members in attendance. Professor R. P. Bailey, Chairman, presided.

The following officers were elected for the year 1958–59: Chairman, Professor M. W. Oliphant, Georgetown University; Vice-Chairmen, Professor Herta T. Freitag, Hollins

College and Professor Joseph Milkman, United States Naval Academy; Secretary, Professor D. B. Lloyd, District of Columbia Teachers College; Treasurer, Professor T. W. Moore, United States Naval Academy.

The following papers were presented at the meeting:

1. *Certain graphical solutions to the heat conduction equation for an insulated, infinite metal slab*, by Mr. W. H. Holter, Atlantic Research Corporation, Alexandria, Virginia.

The solution to the heat equation for the case of a metal slab separated from a transient flow of hot gases by a layer of insulation was presented. Simplified, approximate solutions were obtained at the two boundaries of the insulation. Graphical representations of these solutions provide a rapid method of determining the insulation required to maintain the temperature of the metal slab below a given level. The wide application of the graphs to various heating problems was illustrated.

2. *The solution of ordinary differential equations with constant coefficients by analog computer methods*, by Professor W. L. Fields, Hampton Institute.

The machine methods for addition, subtraction, multiplication, division, and integration were shown. These operations were combined to demonstrate the solution of differential equations.

3. *Advances in automatic programming for digital computers*, by Mr. R. F. Reiss, Babcock and Wilcox Co., Atomic Energy Division, Lynchburg, Virginia, introduced by the Secretary.

The translation of a mathematical problem into use for computer operation ordinarily requires much training and experience. To simplify the programming of such problems and to minimize the personnel training required, methods known as "automatic programming" have been devised. One of the more sophisticated systems, the FORTRAN formula translator, was described in detail and compared with ordinary programming techniques.

4. *Experiments in solving ordinary differential equations*, by Mr. W. Timlake, Babcock and Wilcox Co., Lynchburg, Virginia, introduced by Professor M. G. Humphreys.

Truncation error is usually a dominant factor in the choice of which numerical integrating scheme is applied to a set of ordinary differential equations. This paper presented a clear example of a system of differential equations in which stability (as defined by O'Brien, Hyman and Kaplan), and propagated error completely overshadow the problem of truncation. The difficulty of obtaining a "good" *a priori* bound on Δt was also indicated.

5. *A study of variations in the viewing of a picture*, by Professor Herta T. Freitag, Hollins College, and Mr. A. H. Freitag, Hollins College School, Hollins College, Virginia, presented by Professor Freitag.

The well-known calculus problem of determining the position from which a picture appears "at its best" leads to an interesting consideration of other related factors. Arbitrarily varying the observer's height, his position, the height of the picture, the height at which it is hung—keeping any two of these parameters constant, while varying the others, leads to six cases. In each case the resulting functional relationships are conic sections—parabolas, hyperbolas, or even degenerate conics.

6. *Cardioids and rolling polygons*, by Professor R. C. Yates, College of William and Mary. (By invitation.)

Professor Yates discussed the properties and applications of the cardioid, based upon its fundamental geometric structure. He interpreted the curve as a *roulette*, a *conchoid*, a *pedal*, a *caustic*, and an *envelope of circles*. A unique feature involved the determination of arc length and area by means of rolling polygons. This was based upon formulas for sums of sines and cosines of integral multiples of an angle of roll, sums which have limits as the number of sides of the polygon

increases. He pointed out the generalization of this method to all closed cycloidal curves. The lecture was illustrated with colored drawings, which were distributed to the audience.

D. B. LLOYD, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-fourth annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 19, 1958, at the University of Nebraska, Lincoln, Nebraska, in conjunction with the meetings of the Nebraska Academy of Sciences. Professor Edwin Halfar presided. There were 35 persons in attendance, including 25 members of the Association.

The following officers were elected for 1958–1959: Chairman, Professor J. F. Wampler, Nebraska Wesleyan University; Vice-Chairman, Professor Edwin Halfar, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

The following papers were presented:

1. *The icosahedron*, by Professor W. G. Leavitt, University of Nebraska.

Consider a cylinder with cones of the same radius at both ends, the radius of the cylinder being equal to the distance between the bases of the cones, and the angle between the axis and the slant of the cones being a little less than 60°. Five equally spaced points are marked around the base of each cone in such a way that if P is a point on the upper cone, the point P' directly below P bisects the arc between two of the points marked on the lower cone. These ten points together with the apexes of the cones form the twelve vertices of the icosahedron.

(Analysis of method discovered by a local machinist.)

2. *Integral means and subfunctions*, by Professor L. K. Jackson, University of Nebraska.

Let $f(x)$ be a bounded integrable function defined on an interval. The integral mean $F_h(x)$ is one degree "smoother" than $f(x)$, for example, if $f(x)$ is continuous, $F_h(x)$ is differentiable. Furthermore, the integral-mean operation preserves linear functions and transforms convex functions into convex functions. In this paper is discussed the problem of extending these results to more general differential equations. Specifically, given a differential equation, a smoothing operation is sought which preserves solutions of the differential equation and transforms subfunctions into subfunctions. A solution is given for linear second order differential equations.

3. *Hausdorff separation and compactness*, by Professor Edwin Halfar, University of Nebraska.

The fact that compact subsets of a Hausdorff space are closed suggests the problem of determining restrictions on a space so that the property of being Hausdorff is equivalent to the property that compact subsets are closed. A simple theorem of this type is proved, and an example illustrating the necessity of some such restriction is given.

4. *First Nebraska (Ninth National) Mathematics Contest*, by Professor H. M. Cox, University of Nebraska.

1685 contestants from 130 high schools were enrolled in the Mathematics Contest held March 27, 1958. Scoring was provided by the University of Nebraska's Bureau of Instructional Research. Norms were prepared on 1405 contestants from 121 high schools. (Mimeographed report is available for distribution.)

5. *A report on the use of television for stimulating more interest in mathematics*, by Professor W. E. Mientka, University of Nebraska.

During 1957, "Fun with Figures" was presented in eighteen half-hour programs on KOLO-TV (Reno, Nevada) and in eleven half-hour programs over KUON-TV (University of Nebraska, Lincoln, Nebraska). Each program consisted of three parts: (1) an introduction to some mathe-

matical problem by means of a model (Mobius strip, curves of constant breadth, towers of Hanoi, *etc.*); (2) a discussion of various mathematical topics, not ordinarily considered in usual mathematics courses (tests for divisibility, the conic sections, the Königsberg bridge problem, *etc.*); (3) a problem of the week. The verbal and written response indicated that the objectives of the series were achieved.

6. *Types of errors in Dickson's "History of the Theory of Numbers,"* Vol. 2, by Mr. H. W. Becker, Radio Engineering Institute, Omaha, Nebraska.

Typical of the three hundred errors discovered are the failure to tell a triangle from a tetrahedron (p. 210), whether two problems are the same, or different (p. 194 and p. 210), whether two solutions are the same or different (p. 210, p. 212, p. 213), to recognize a general solution (all three references), and to see through a fallacious proof even when only a few lines long (p. 502).

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The forty-second annual meeting of the Ohio Section of the Mathematical Association of America was held at Denison University, Granville, Ohio, on Saturday, April 26, 1958. Professor Sam Selby, Chairman of the Section, presided at the morning and afternoon sessions. There were 117 persons registered in attendance, including 97 members of the Association.

Officers elected for the coming year are: Chairman, Professor L. E. Bush, Kent State University; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor Melvin Bloom, Miami University. The members of the Program Committee are: Professor H. E. Tinnappel, Bowling Green State University, Chairman; Professor R. W. Shoemaker, University of Toledo; Professor C. W. Topp, Fenn College.

The following papers were presented:

1. *Critique on preparation of elementary and secondary school teachers of mathematics*, by Professor Nathan Lazar, The Ohio State University. (By invitation.)

2. *Advanced slide rule techniques* (*cubic equations, addition and subtraction*), by Dr. B. L. Schwartz, Battelle Memorial Institute, Columbus, Ohio.

a. Any cubic equation is readily reduced to the form $y^3+Py=Q$. This equation can be solved with one setting of any slide rule having the usual *A*, *B*, *C*, and *D* scales.

b. Normally there is no special advantage in performing addition on a slide rule. However, there may be one when a formula combines additions with normal slide rule operations, *e.g.*, multiplications. The method presented requires as many operations as a multiplication followed by a division. For use with rules with other scale arrangements than the *K* & *E* series 4080, the formula the process solves is given by: $u+v=[\sqrt{u}/\sin \text{arc tan } (\sqrt{u}/\sqrt{v})]^2$.

3. *The modern approach to LR²H*, by Professor R. F. Rinehart, Case Institute of Technology.

How writers of literature might impart new austerity—and hence beauty—into their works by employing modern mathematical language was illustrated by discussion of the story of Little Red Riding Hood (LR^2H) in modern mathematical parlance. For example, the grandmother, *G*, could be denoted by $f(f(LR^2H))$, where f is the Murrow mapping (*i.e.*, person-to-person) "daughter" into "mother."

4. *The derivative as a front of expansion*, by Professor Wayne Dancer, University of Toledo.

To give the derivative meaning within the students' experience let us visualize it as a "front of expansion" along which a geometric figure grows into a larger but similar figure. A square of area x^2 grows into a larger square by expansion along two sides, $2x$, and a circle expands along the

circumference, $2\pi r$. These are the derivatives of the variable areas. The standard "area under a curve," bounded laterally by a fixed ordinate and a variable ordinate, can expand only along the variable ordinate, *i.e.*, $dA/dx = y$. Similarly, the front of expansion of a variable sphere is the surface of the sphere, expressed: $d(4\pi r^3/3)/dr = 4\pi r^2$. A right circular cylinder with variable radius and altitude has two fronts of expansion: the top, πr^2, and the lateral surface, $2\pi rh$. These are the partial derivatives of the volume with respect to h and r respectively.

5. *The new Dartmouth mathematics curriculum*, by Professor J. L. Snell, Dartmouth College. (By invitation.)

A mathematics curriculum in a liberal arts college must fulfill a variety of needs. It must provide an opportunity for the average liberal-arts student to learn of the nature of advanced mathematics. It must provide specialized training for various scientists who will use mathematics as a tool. It must make provision for the man who wants to specialize in mathematics, whether he be of the caliber of a creative mathematician or of average ability. And finally, it must provide information to the entire undergraduate body on modern mathematics as a living and creative field, with applications in the physical, biological and social sciences. The speaker described the way in which Dartmouth College has attempted to fulfill these goals.

6. *Report from the Commission on Mathematics of the College Entrance Board*, by Professor E. R. Ranucci, Newark State College, introduced by the Secretary. (By invitation.)

The Commission on Mathematics of the College Entrance Examination Board grew out of the concern of the Mathematics Examiners that the Board's tests were not reflecting fully and appropriately the emerging programs of mathematics instruction in forward-looking college preparatory schools, both public and private, and moreover that the standard curriculum taught in most secondary schools was sadly out of date.

The Commission accepts the following theses: (1) In the past thirty years the nature of mathematics as a subject has been substantially altered by the results of mathematical research. (2) Despite these developments, school and college mathematical curricula have been largely unaltered. (3) It is both possible and desirable to rectify this situation.

7. *Independent studies at Oberlin College*, by Professors John Baum and Wade Ellis, Oberlin College, given by Professor Ellis.

Four of Oberlin College's eight sections of regular freshman mathematics were organized, in 1957–1958, as two control and two experimental independent study classes. Professors Baum and Ellis each taught one control and one experimental section. The control sections went through the usual course experience. The experimental sections attended classes from September 17 through December 14 (about 13 weeks) and from April 8 through May 24 (about 7 weeks), except for one week in November. During the remainder of the school year, they were out of contact with their instructors.

Tests administered on April 8, and covering material studied during the out-of-class period by both experimental and both control groups, failed to reveal any significant difference between the two types of classes insofar as mastery of subject matter and facility with techniques were concerned.

8. *The "Cincinnati Experiment,"* by Professor Gaylord Merriman, University of Cincinnati.

The speaker reported the fate of three experimental courses which had been outlined previously (Oberlin meeting, 1956): (a) a course, for secondary school teachers, in modern concepts being urged for inclusion in the high school programs; (b) a course of the Kemeny-Snell-Thompson type for majors in certain social sciences; (c) a course of tutorial readings primarily for advanced placement students. All three had been successful. The first two are now integrated into the regular

program. The last is being abandoned because of large curricular changes which accommodate students into a more flexible program. The talk concluded with a detailed outline of this new curriculum—a continuing "experiment."

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The semi-annual meeting of the Oklahoma Section of the Mathematical Association of America was held at Central State College, Edmond, Oklahoma, on Saturday, April 19, 1958. Professor W. A. Rutledge, Chairman of the section, presided. There were 52 persons in attendance, including 41 members of the association.

Several committee reports presented at the business meeting are summarized with the abstracts of the papers presented. The section voted that the sum of fifteen dollars be given to the Committee on the Oklahoma Visiting Lecture Service from the section treasury. The committee was instructed to determine the type of lecture program desired by Oklahoma and Arkansas high schools, if any, and if possible, to implement the program before the fall meeting. It was further voted that the committee be specifically instructed to secure lecturers, and that both the lecturers and their topics be subject to the approval of the committee.

The following papers and reports were presented:

1. *Some properties of connectivity maps*, by Professor O. H. Hamilton, Oklahoma State University of Agriculture and Applied Science.

According to Nash, a connectivity map from a space A into a space B is a transformation T such that the induced transformation g of A into $A \times B$ defined by $g(p) = p \times T(p)$ transforms connected subsets of A onto connected subsets of $A \times B$. Examples of connectivity maps which are not continuous were given and some simple properties of connectivity maps were derived and compared with different characterizations of continuous transformations.

2. *Two theorems with respect to sequences of Banach spaces*, by Mr. Robert Welland, student, University of Oklahoma, introduced by the Secretary.

The norm for a Banach space often determines the set of points to be in the space. This situation leads one to ask how the spaces vary if the norm is changed. Unions and intersections of ascending and descending sequences of Banach spaces were considered.

3. *Evidence that* $2^{8191}-1$ *is prime*, by Mr. J. W. Sehestedt, Hoyt, Oklahoma.

The purpose of this paper is to present evidence that $2^{8191}-1$ is prime, as conjectured by E. Catalan. The evidence is dependent on facts verifying that a Mersenne prime used as P value in his formula will always produce a perfect number. It can be shown that $2^{8191}-1$ has characteristics identical to all known Mersenne primes, in relevance to perfect numbers. Computation is greatly reduced and simplified by using the sequence of a few digits at the end of the number to identify it. A graphic presentation in which all numbers are positionized was also demonstrated.

4. *Certain nonaffine projective transformations in 4-space*, by Professors Simon Green and W. A. Rutledge, University of Tulsa, presented by Professor Green.

This paper is an extension of the work done by J. S. Taylor, St. Kwietniewski, K. Kommerell and others on the problem of representation of equations in two complex variables by surfaces in a real 4-space. The problem of determining the nonaffine projective transformations in 4-space that leave invariant the set of regular, tangent planes to a surface is studied. It is found that all such transformations are degenerate.

5. *Restriction of acreage with allowance for variation of productive capacity*, by Mr. K. C. Cartwright, Vandervoort, Arkansas.

When farm acreage was restricted to limit production, the farmer utilized his labor and better

methods on the lesser area to the effect of getting more production than he did before, nullifying the assignment of acreage. The method of reduction given is linked with the best production that has been made in the area. This makes it difficult for even one farmer to excel this best record. For all of them to excel it is virtually impossible. The remarks deal with the application of the same type of function to the initial aspect of the solubility of gases in liquids.

6. *Summability operators in topological groups*, by Mr. John Thomas, student, University of Oklahoma.

Summability operators are defined in a first-countable, Hausdorff topological group and a set of 3 sufficient conditions given for an operator to be regular. It is shown that 2 of the conditions are always necessary, and the third is also, where G is the additive group of a Banach space over a field complete in some (not necessarily Archimedean) valuation. When the Banach space is a complete field, the theorem is shown to be equivalent to the classical Toeplitz theorem and its recent extension to non-Archimedean fields.

7. *Some optimization problems related to rocket propulsion*, by Mr. G. M. Ewing, U.S.A. Artillery and Missile School, Fort Sill, Oklahoma. (By title.)

The paper formulates a number of maximum and minimum problems involving the choice of three time-functions termed the thrust program, the thrust attitude program, and the staging program, each subject to appropriate restrictions. Attention is confined to the simplest mathematical model, *viz.* the point mass, flat stationary earth, constant gravity idealization. The problems are related to, but distinct from, classical nonparametric problems of the calculus of variations. The only results known to the author are existence theorems together with complete solutions for trivial cases.

8. *The cross ratio in n-dimensional space*, by Mr. Robert Strong, student, University of Oklahoma, introduced by the Secretary.

By considering one form of the cross ratio one arrives at a generalization to a cross ratio of four points with respect to $n-1$ points in an n-dimensional space. From this expression one can show the equivalence of the definition of the cross ratio of four hyperplanes of a pencil and of the definition given by N. A. Kolmogorov in terms of hypervolumes.

9. *A dispersion concept pertaining to the damage potential of an artillery projectile*, by Mr. O. S. Spears, U.S.A. Artillery and Missile School, Fort Sill, Oklahoma.

When the vulnerable area of a target is known, the density and angular distribution of effective fragments about a bursting projectile are used to determine the probability of required damage from various positions. If $g(x, y)$ denotes the probability of damage to a target centered at the point (x, y), then a measure of effectiveness ϕ is defined as follows: $\phi = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x, y)dxdy$.

While these integrals have the dimensions of area, they are ordinarily not discussed with respect to any particular size and shape. On the other hand, when a number of projectiles are fired into a specific target area, it is often necessary to consider some specific configuration attached to the numbers ϕ. A simple approach to this matter is presented in this paper.

10. *Desmic systems of tetrahedrons*, by Professor N. A. Court, University of Oklahoma. (To be published in this MONTHLY under Mathematical Notes.)

11. *Report from Committee on Oklahoma Visiting-Lecture Service*, by Professor Katherine Mires, Northwestern State College, Chairman.

The committee has considered several possible types of visiting-lecture programs suitable for high schools, but found itself stopped from further action by lack of time and funds. It is hoped the high school lecture service will have been implemented before the October meeting of the section.

12. *Report from the Committee on High School Mathematics Contest*, by Professor H. N. Carter, University of Tulsa, Chairman.

Although the committee was late in forming, and met with several difficulties, the contest was held, thanks to the assistance of Professor Fagerstrom.

13. *Report from the advisory committee of Junior Academy of Science*, by Professor R. B. Deal, Oklahoma State University of Agriculture and Applied Science.

Over 1900 science and mathematics teachers in Oklahoma have been invited to present academic papers (in contrast with the usual "model" or "gadget" used in science fairs). Forty-seven papers were submitted for consideration, from which the program for the May 10th meeting will be selected.

R. V. ANDREE, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, New Mexico, on April 11–12, 1958. Dr. R. C. Hildner, Chairman of the Section, presided at the afternoon session on April 11, and also at the morning session on April 12. There were 63 persons in attendance, including 47 members of the Association.

The following officers were elected: Chairman, Professor J. H. Butchart, Arizona State College at Flagstaff; Vice-Chairman, Professor R. B. Crouch, New Mexico College of Agriculture and Mechanic Arts; Secretary-Treasurer, Professor Deonisie Trifan, University of Arizona.

The following papers were presented:

1. *A note on cancellation of groups of rank one*, by Professor E. A. Walker, New Mexico College of Agriculture and Mechanic Arts.

This note is concerned with the following question: if G and H are Abelian groups, if F is an Abelian group of rank one, and if $F \oplus G \cong F \oplus H$, is $G \cong H$? It has been shown that if F is infinite cyclic then $G \cong H$ (E. A. Walker, Proc. Amer. Math. Soc., Oct. 1956). Also if F is isomorphic to the additive group of rational numbers, then $G \cong H$. For arbitrary groups of rank one the following hold: (a) G is isomorphic to a subgroup of H and H is isomorphic to a subgroup of G; (b) If G is torsion free and almost locally pure (for definition of almost locally pure, see abstract 542–108 in the Notices of the Amer. Math. Soc.) then $G \cong H$.

2. *Problems in diophantine approximation*, by Professor G. M. Petersen, University of New Mexico.

Let $\{s_n\}$ be a sequence for which $0 < s_n < 1$ for all n, and I any interval (a, b) in $(0, 1)$. If $I(x) = I$, $x \in (a, b)$, and $I(x) = 0$, $x \notin (a, b)$ then the sequence $\{s_n\}$ is well distributed if and only if $\lim_{p \to \infty} [I(S_{+n1}) + \cdots + I(S_{n+p})]/p = b - a$ uniformly in p, for all intervals (a, b). This definition is a modification of that of Weyl for uniformly-distributed sequences. Different cases of when uniformly-distributed sequences are also well distributed are discussed.

3. *Cascading solutions of a nonhomogeneous equation of Mathieu-Hill type*, by Professor R. M. Conkling, New Mexico College of Agriculture and Mechanic Arts.

The differential equation $\ddot{\theta} + A\dot{\theta} + (B - C \cos \omega t)\theta = D \cos \omega t$ becomes a nonhomogeneous Mathieu equation when the first derivative is removed by the usual substitution. Upon replacing $\cos \omega t$ by a square wave $S(t)$, the method of Louis Pipes (J. App. Phys. vol. 24, July 1953, pp. 902–910) can be adapted to give the solution as a matrix equation.

4. *Government research*, by Professor Charles Wexler, Arizona State College, Tempe.

This was a proposal that the government gradually disband its own groups doing theoretical research not requiring specialized and extensive equipment, and parcel out the research on a part-time contract basis to appropriate college teachers. The following advantages are seen: (1) An influx of qualified technical personnel into college teaching where they are sorely needed. Their teaching salaries would be supplemented by the part-time contracts. (2) A better intellectual atmosphere to work in, unhampered by civil service restrictions and red tape. (3) More first-class minds contributing to the research. (4) As much or more research as is done now, at less cost.

5. *The ubiquitous integral equation of Abel*, by Dr. G. M. Wing, Los Alamos Scientific Laboratory.

The integral equation $(*)F(t)=\int_0^t(t-s)^\mu G(s)ds$, $\mu>-1$, and its generalizations are considered. Various classical instances in which (*) arises are discussed. Two recent problems at the Los Alamos Scientific Laboratory have led to equations of type (*). One has to do with an x-ray technique for determining the density ρ of material in a sphere, where ρ depends upon the radial position only. The other is an attempt to determine whether the neutrons observed in an experiment related to the controlled thermonuclear reaction arise from such a reaction or whether they have a spurious source.

6. *Some remarks on splitting theorems for monomial groups*, by Professor R. B. Crouch, New Mexico College of Agriculture and Mechanic Arts.

Let H be a group, U a set and $\Sigma(H; B, d, C)$ the monomical group of H over U. Here $o(U)=B$ and $d\leqq C\leqq B^+$. The normal subgroups of Σ are known (R. B. Crouch and W. R. Scott, *Normal Subgroups of Monomial Groups*, Proc. Amer. Math. Soc., vol. 8, 1957, pp. 931–936). Let N be a normal subgroup of Σ contained in the basic group V. The question then arises, does Σ split over N? Denote by G the subgroup of H from which the factors of multiplications of N may be chosen. Let the subgroup G_2 of products of factors of multiplication of N be G. Assume that H splits over G and $H=GK$. It can then be shown that Σ splits over N, $\Sigma=NR$ and R is the weak direct product of K unioned with the symmetric group $S(B, C)$.

7. *A set theoretic counterexample*, by Dr. W. W. Bledsoe, Sandia Corporation, introduced by the Secretary.

Let F be a family of sets and $S=\bigcup B\in FB$. DEFINITION. *Borel F = The smallest σ-field containing F.* Due to A. P. Morse is the following DEFINITION. *Bor $F=EA$ (A is ϕ-measurable whenever ϕ measures S and $F\subset$ measurable ϕ).* THEOREM. *Borel $F\subset$ Bor F.* Well-known is THEOREM A. *Borel $F=\bigcup G\subset F$; G is countable Borel G.* It is shown by counter-example that the result of Theorem A no longer holds whenever "Borel F" is replaced by "Bor F."

8. *Determination of atmospheric density from satellite observations*, by Dr. Herbert Knothe, Holloman Air Development Center, introduced by the Secretary.

9. *Properties of the tractrix and catenary*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

Using synthetic methods, Professor Butchart showed that the catenary is the evolute of the tractrix, that the tractroid of revolution has constant negative total curvature, that the catenoid of revolution has less area than any other surface joining two circles on the same axis, and that this surface has zero mean curvature. He also obtained the derivatives of the inverse trigonometric functions directly from diagrams and showed that the locus of the focus of a parabola rolling on a line is the catenary.

10. *Mathematical problems arising in engineering analyses of heat transfer processes*, by Dr. Knox Millsaps, Holloman Air Development Center, Holloman, New Mexico.

The mathematical problems that arise in the solution to the partial differential systems which

describe the heat transfer from a solid boundary to a moving fluid are outlined. The particular case of the heat transfer to Hagen-Poiseuille flows is given in some detail as a typical example of the mathematical analysis. Possible generalizations to other physical situations are also discussed.

11. $\sum(H, A, C, D)$ *does not split over* $\sum(H, A, D, d)$, by Mr. A. B. Gray, Jr., New Mexico College of Agriculture and Mechanical Arts, introduced by the Secretary.

It is well known that $\sum(H, A, D, d)$ is a normal subgroup of $\sum(H, A, C, B)$ with $D \leqq C$, $d < B$. The group $S(A, B)$ does not split over $S(A, d)$. (This result was communicated to me by W. R. Scott.) If $\sum(H, A, C, B) = \sum(H, A, D, d) \cup L$ and $\sum(H, A, D, d) \cap L$ is the identity then let T be the set of all s such that vs is in L where v is a multiplication with less than C nonidentity entries. The set T is a subgroup of $S(A, B)$, $S(A, B) = S(A, d) \cup T$ and $S(A, d) \cap T$ is the identity. Thus $S(A, B)$ splits over $S(A, d)$ contradicting the above so $\sum(H, A, C, B)$ does not split over $\sum(H, A, D, d)$.

12. *Science, society, and survival*, by Dr. J. W. McRae, President of the Sandia Corporation.

13. *Loci associated with families o plane osculants*, by Mr. Louis Child, New Mexico College of Agriculture and Mechanic Arts.

At P_0 of a plane curve Γ, a member f_4 of the 5-parameter family F_4 of 4-pointic cubics meets each member γ_4 of the 1-parameter family C_4 of 4-pointic conics in 4 points at P_0 and ordinarily at 2 additional points P_1, P_2 whose abscissas x_1, x_2 are the roots of $\psi_0 x^2 + 2\psi_1 x + \psi_2 = 0$ where the ψ_i are polynomials of degree 4 in λ, the parameter of C_4. The roots $\lambda_1, \cdots \lambda_8$ of $D = \psi_0\psi_2 - \psi_1^2 = 0$ then determine points $P_1, \cdots P_8$ at which f_4 doubly contacts the corresponding conics. The Halphen point $H(f_4)$ for $P_1, \cdots P_8$ is thus unique for f_4. The set $H(F_8)$ of all such Halphen points for the members f_8 of the 1-parameter subfamily F_8 of 8-pointic cubics forms a Halphen curve associated with P_0, and the point $H(f_9)$ on this curve corresponding to the ordinarily unique 9-pointic cubic is a second Halphen point associated with P_0.

14. *Some formulas related to least squares smoothing and predicting formulas*, by Mr. W. L. Shepherd, Texas Western College.

Consider the points $(0, z_0), (1, z_1), \cdots$. By suitable translation we have

$$(1) \qquad V\sum_{i=0}^{p-1} z_{i+k} = m_k \sum_{i=0}^{p-1} i + b_k \cdot p, \qquad (2) \qquad \sum_{i=0}^{p-1} z_{i+k} \cdot i = m_k \sum_{i=0}^{p-1} i^2 + b_k \sum_{i=0}^{p-1} i$$

as normal equations for the linear least squares fit to the p points $(k, z_k), \cdots, (p+k-1, z_{p+k-1})$, yielding a predicted value, say $\bar{z}_{p+k}$, for z_{p+k}, given by $\bar{z}_{p+k} = m_k p + b_k$. The formulas

$$(3) \qquad \sum_{i=0}^{p-1} z_{i+k+1} = \sum_{i=0}^{p-1} z_{i+k} + z_{k+p} - z_k,$$

$$(4) \qquad \sum_{i=0}^{p-1} z_{i+k+1} \cdot i = \sum_{i=0}^{p-1} z_{i+k} i + p \cdot z_{k+\bar{p}} - \sum_{i=0}^{p-1} z_{i+k+1}$$

are derived and used to obtain $\bar{z}_{p+k+1}$. (3) and (4) are generalized.

15. *Finite divisibility in Abelian groups*, by Professor D. L. Boyer, New Mexico College of Agriculture and Mechanic Arts, introduced by the Secretary.

Some comments on a paper presented to the American Mathematical Society (abstract number 542-108) by D. L. Boyer and E. A. Walker.

16. *An iterative method for determining a differential equation*, by Dr. J. A. Ward, Holloman Air Force Base, Holloman, New Mexico.

The purpose of this paper is to give an iterative method for finding the coefficients of a differential equation if the form of the equation and some points on its solution are known.

The procedure applies for variable or constant coefficients. The method was developed on a digital computer for "real time" computation and for data that includes "noise."

17. *On a decomposition of convex sets due to N. Aronszajn*, by Professor K. R. Lucas, New Mexico College of Agriculture and Mechanic Arts.

18. *On the three-choice problem in dynamic programming*, by Professor I. I. Kolodner, University of New Mexico.

Let for $i=1, 2, \cdots m$, $j=1, 2, \cdots n$: (1) $0 \leqq p_i < 1$, (2) $0 \leqq c_{ij} \leqq 1$, $c_{ij}=1-c_{ij}$, (3) $x_j \geqq 0$, (4) $T_i[f]=p_i(\sum_j c_{ij}x_j+f(c_{i1}x_1, c'_{i2}x_2, \cdots, c'_{in}x_n))$, (5) $T[f]=\max_i T_i[f]$. The optimal expected payoff $f(x_1,x_2, \cdots, x_n)$ of an m-choice discrete decision process is the solution of $f=T[f]$. R. Bellman showed that this equation has a unique solution which is continuous but not necessarily differentiable. For $m=2$, $n=2$, $c_{12}=c_{21}=0$ (solution f_3) a satisfactory resolution method has been found. THEOREM. *Let* $a_{ij}=p_ic_{ij}(1-p_ic'_{ij})^{-1}$, $\bar{a}_j=\max_i a_{ij}$, $a_j=\min_i a_{ij}$, $\bar{f}=\sum_j \bar{a}_jx_j$, $f=\sum_j a_jx_j$. Then: (1) $f=\lim T^k[\bar{f}]=\lim T^k[f]$, furthermore, the first (second) sequence is antitone (isotone) and $f \leqq f \leqq \bar{f}$; (2) f is an increasing function of the p_i; (3) if for some s, $\bar{a}_j=a_{sj}$ then $f=\bar{f}$; (4) for $m=3$, $n=2$, $c_{12}=c_{21}=0$ (three-choice problem, solution f_3), $f_2 \leqq f_3 \leqq \bar{f}$.

D. TRIFAN, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The annual spring meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 3, 1958 at Washington and Jefferson College, Washington, Pennsylvania. The Section Chairman, Professor I. Dee Peters of West Virginia University, presided at the morning session. Professor W. G. Brady of Washington and Jefferson College presided at the afternoon session. There were 64 persons registered, including 28 members of the Association.

At the business meeting, the by-laws of the section were amended to define the geographical boundaries of the section and to provide for biennial election of officers. Professors E. E. Posey and J. H. Neelley, co-chairmen of the Section Committee on High School Contests, reported that 4220 students, representing 168 of the 400 high schools invited, participated in the 1958 contest. Professor J. K. Stewart of West Virginia University described their plans for a summer 1958 institute for high school teachers. It is intended to teach specific subject matter in mathematics and in physical science.

The following short papers were presented:

1. *Some remarks on derivations of Lie algebras*, by Professor G. F. Leger, University of Pittsburgh, introduced by Professor J. C. Knipp.

This paper gives a discussion of some results on derivations of Lie algebras and culminates in an example of a Lie algebra L with Levi decomposition $L=S+R$ such that $D(L)$ splits over $I(L)$ but $D(R)$ does not split over $I(R)$. This settles a question left open by the author in an earlier note (*Proc. Amer. Math. Soc.* vol. 4, 1953, pp. 511–514).

2. *Inequalities on the gravest tone of a vibrating string*, by Professor R. A. Moore, Carnegie Institute of Technology.

The lowest eigenvalue of the differential system $y''+\lambda p(x)y=0$, $y(0)=y(a)=0$, is discussed. Upper and lower bounds for λ_1 are obtained for positive continuous functions $p(x)$, normalized by $\int_0^a p(x)dx=w$, in each of the following classes: (i) $p(x) \leqq m^2$, (ii) $p(x)$ concave, (iii) $p(x)$ convex. In case (i) the result is a generalization of Liaponnoff's inequality: $\lambda_1 \geqq 4/(aw-w^2/2m^2)$.

3. *Some algebraic, analytical and topological properties of algebraic curves*, by Professor Mario Benedicty, University of Pittsburgh, introduced by Professor J. C. Knipp.

Exposition of some modern developments of the most important branches of algebraic geometry, such as theory of equivalence on an algebraic manifold, fields of algebraic functions, abelian functions and group varieties, and their relationships with classical properties of algebraic

curves, in particular of elliptic cubics, such as linear series, algebraic functions of a point, elliptic integrals, birational transformations of the curve onto itself.

4. *What must be done about a new type of freshman*, by Professor J. H. Neelley, Carnegie Institute of Technology.

This paper opposes the teaching of analytic geometry and calculus by high schools. It points out that there is no credit that can be given by colleges for such courses; that many high school teachers are not properly prepared to teach these subjects; that most failures in freshman mathematics can be traced to algebraic ignorance. Hence, it advises more algebra instead of college courses in high school.

5. *Airy's Differential Equation and Airy Functions*, by Professor William Laird, University of Pittsburgh.

A brief historical account of the researches of Sir G. B. Airy is presented together with the investigations leading to certain improper integral functions which satisfy the equation $y''-xy=0$. Properties of these solutions, called Airy Functions, are described together with a discussion of the differential equation and its application.

6. *An inverse Sturm-Liouville problem*, by Professor A. D. Martin, Carnegie Institute of Technology.

The *focal function* $f(a, \lambda)$ of a Sturm-Liouville equation $(ry')'+py=0$ is defined to be the first zero to the right of a of the unique solution $y=u(x, \lambda)$ such that $u(a, \lambda)=1$ and $r(a)u'(a, \lambda)=-\cot\lambda$. It is then proved that if two equations $(r_iy')'+p_iy=0$, r_i, p_i continuous functions on $(-\infty, \infty)$ while $r_i>0$ there for $i=1, 2$, have focal functions f_1 and f_2 respectively such that for each x, $f_1(x, \lambda)=f_2(x, \lambda)$ for a sequence of values of λ which tend to zero, then $r_1(x)\equiv r_2(x)$ and $p_1(x)\equiv p_2(x)$.

7. *Experimental science, mathematics, and the undergraduate*, by Dr. D. H. Shaffer, Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

It is becoming increasingly important for the worker in the physical sciences and engineering to recognize the existence of variability and chance occurrences in experimental work and to learn of the methods available for making sensible inferences from data in their presence. For such an awareness to be realized, the basic concepts and techniques of statistics must be taught (1) as a branch of mathematics, (2) to all students of science and engineering, and (3) as early as possible in the academic program. The teachers of mathematics should assume the responsibility of leading these students in this fascinating but underemphasized science.

B. H. MOUNT, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-seventh annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois College, Jacksonville, Illinois, on May 8 and 9, 1958. Professor C. T. McCormick, Chairman of the Section, presided at all sessions. There were 66 persons in attendance, including 50 members of the Association.

At the business meeting on Friday afternoon the following officers were elected to serve for the coming year: Chairman, Professor A. E. Hallerberg, Illinois College; Vice-Chairman, Professor B. K. Brown, Milliken University; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University.

Following the brief welcome by Professor E. B. Miller of Illinois College, the following program was presented:

1. *A study of the difficulties in algebra*, by Professor G. H. Miller, Western Illinois University, introduced by the Secretary.

This study was based on the results of an analysis of survey forms given to 389 students in two universities and two junior colleges in California. The recommendations based on the con-

clusions of the study were: (1) That more time be allotted to the topic of exponents and their interelationships with related topics in algebra such as logarithms, annuities, binomial expansion, and progressions; (2) the development of better teaching techniques for the difficult topics in algebra, *i.e.* probability and statement problems; (3) the suggestion that a study be made to determine the most vital topics for instruction of higher mathematics and science.

2. *A letter of Philip Melanchthon*, by Professor Marian Moore, Southern Illinois University.

Philip Melanchthon, 16th century educator and religious reformer, wrote many texts, edited many classics, and furnished introductions to others. In particular he wrote an introduction to the 1537 Basle edition of Euclid, which was a reissue, with slight changes, of the 1516 Paris edition. Melanchthon's introduction, *A Letter to Young Students*, is chiefly a plea for humanism in education. The notion of an axiomatic system being chosen for its usefulness was evidently unknown to Melanchthon, who apparently agreed with an ancient view that geometry was handed down from Olympus.

3. *The use of semigroups in the teaching of modern algebra*, by Professor A. O. Lindstrum, Jr., Knox College.

The author defined a semigroup and illustrated it by using a mapping of a set into itself so that many important and fundamental concepts such as the associative law, the commutative law, and the cancellation law could be made clear. The use of semigroups in defining factoring, in stating multiplicative properties of ideals, and in constructing linear associative algebras was also briefly discussed.

4. *Peeks behind the Curtain*, by Professor S. S. Cairns, University of Illinois.

This nonmathematical after-dinner speech compared Soviet with American educational philosophies, problems and procedures.

5. *UICSM course in deductive plane geometry*, by Professor H. E. Vaughan, University of Illinois.

The paper dealt with a set of postulates adequate to a rigorous development of Euclidean plane geometry.

6. *Evaluation of National Science Foundation Summer Institutes*, by Professor M. Anice Seybold, North Central College.

The author recounted some of her experiences as an evaluator of National Science Foundation Summer Institutes in Science and Mathematics during the summer of 1957. She discussed physical facilities, students, teachers and courses, illustrating how they varied in the institutes she visited. New kinds of courses and new kinds of teaching are evolving as a result of these institutes. There is some indication that a new master's degree in the teaching of subject matter may develop.

7. *An experiment with modern mathematics*, by Professor Flora Dinkines, University of Illinois, Chicago Undergraduate Division.

The paper presented a report of noncredit lectures provided twice a week in modern mathematics.

8. *A new approach to the teaching of intermediate mathematics*, by Professor Karl Menger, Illinois Institute of Technology.

Summer Institutes initiating high school teachers into concepts of modern algebra and ideas of topology undoubtedly are stimulating experiences to many participants. But it is not apparent how the teachers might extensively utilize those ideas in their own teaching of intermediate mathematics. The main difficulties that beset their students concern the antiquated conceptual and

symbolic frame in which intermediate mathematics and its applications are being presented: the discrepant meanings of the term *variable;* the anonymity of the identity function; the lack of articulate rules for the use of important symbols; and, especially, the inconsistent uses of the letters x and y. What high school teachers ought to be trained in is, above all, the clarification of those obscure ideas.

A. W. McGAUGHEY, *Secretary*

MAY MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at the University of Missouri, Columbia, Missouri, on Saturday, May 3, 1958. Professor J. D. Elder, Vice-chairman of the Section, presided at the morning session, and Professor H. D. Brunk, Chairman of the Section, presided at the afternoon session. In attendance were 52 persons, including 34 members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Francis Regan, St. Louis University; Vice-Chairman, Professor H. M. MacNeille, Washington University; Secretary-Treasurer, Professor Louise Beasley, Lindenwood College. The recommendation of the Committee on High School Contests, namely, that the Missouri Section initiate and conduct the M.A.A. High School contests in the high schools of Missouri in 1959, was voted upon favorably.

The program was as follows:

1. *Convergence of a certain continued fraction*, by Professor David Dawson, University of Missouri.

H. von Koch (*Ofersigt af Kongl. Vetenskaps-Akad. Forhandlinger*, vol. 52, 1895) showed that the continued fraction $1/1+a_1/1+a_2/1+\cdots$ converges in case infinitely many of the a_p are distinct from zero and $\sum a_p<1$. Dennis and Wall (*Duke Math. J.*, vol. 12 (1945), pp. 255–273) improved this result by use of some general convergence theorems. The purpose of this paper is to show that the continued fraction converges in case $\sum a_p \leqq 1$ and three of the a_p are distinct from zero. The proof is elementary, involving only the continued fraction algorithm. If $a_1=a_2=-\frac{1}{2}$, $a_p=0$ for $p>2$, then the continued fraction does not converge.

2. *A natural metric group associated with a metric space*, by Professor J. W. Riner, St. Louis University.

Let (X, d) be a metric space and let H be the free group with free basis X satisfying the relations $xy=yx$ and $x_2=1$ for all x, y in X. Let G be the set of all elements in H of the form: $x_1x_2\cdots x_{2n}=u$. Then G is a subgroup of H. Let $N(u)=N(x_1x_2\cdots x_{2n})=\min. \sum_{i=1} d(x_{v_{2i-1}}, x_{v_{2i}})$ where $(v_1, \cdots, v_{2n})$ is a permutation of the integers $(1, \cdots, 2n)$ and the min. is taken over all such permutations. The function $\partial(u, s)=N(us)$, for u and s in G, is a metric making G a metric group. X is homeomorphic to a subset of G, and if X is connected, then G is connected.

3. *Summer institutes*, by Professor C. A. Johnson, Missouri School of Mines and Metallurgy.

The National Science Foundation Summer Institute program for high school teachers has grown rapidly since its inception in 1953. The Missouri School of Mines Institute for mathematics, physics, and chemistry teachers represents a typical program for improvement of high school science teaching. The Institute gives teachers an opportunity to learn implications of developments in modern mathematics for the secondary program. Teachers who participate in the Institute will be in a better position to carry out the revision in curricular offerings suggested by the Commission on Mathematics.

4. *Nonassociative algebras*, by Professor L. A. Kokoris, Washington University.

The author's recent work on nodal noncommutative Jordan algebras is used to illustrate some concepts of nonassociative algebras.

5. *A representation symbol applied to Waring's theorem, modulo p*, by Professor J. D. Elder, St. Louis University.

The author gave an expository account of the representation symbol, $[a, b, c]$, introduced by Sr. M. F. Torline, C.S.J. in her doctoral dissertation. Implicative properties were given, and their uses in problems connected with Waring's theorem were discussed.

6. *Electronic computers, information and education*, by Professor P. C. Hammer, University of Wisconsin. (By invitation.)

If it is agreed that a proof is in a chain of symbols, machines can and do prove theorems. More generally, they answer questions. The principal role of computing machines is to prove propositions and answer questions. It is an obstacle to the use of machines that mathematicians consider they are dealing with infinite processes, which they say cannot be done by machine. While it may be debated whether there are infinite processes, there is no doubt that no one deals with any process with infinite means. Hence, there are no stated proofs which could not be duplicated on a computer.

MARY L. CUMMINGS, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-first annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado State College, Greeley, Colorado, on Friday afternoon and evening and Saturday forenoon, May 9 and 10, 1958. Professor D. O. Patterson, Chairman of the Section, presided at all three sessions. On Saturday morning the Section held a joint meeting and luncheon with the Colorado Council of Teachers of Mathematics.

There were 107 persons registered for the meeting, including 67 members of the Association. Officers elected at the meeting for 1958–1959 were: Chairman, Professor N. C. Hunsaker, Utah State Agricultural College; Vice-Chairman, Professor J. W. Ault, United States Air Force Academy; and Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *Solving boundary value problems by use of Green's function in conjunction with the Laplace transform and separation of variables*, by Professor L. C. Barrett, South Dakota School of Mines.

The primary purpose of this paper is to point out how an influence function, *i.e.* Green's function, may be utilized together with the Laplace transform and separation of variables to facilitate a solution of boundary value problems of engineering and physics. Among the notable features of the method are: (a) Its capacity to yield the inverse of certain Laplace transforms without requiring recourse to complex variable theory. (b) The method enables one to escape the tedium of the step-by-step procedure, and subsequent use of superposition, usually followed in solving such problems by separation of variables. (c) Time-dependent boundary conditions present no special difficulty to the method.

2. *The radiation of waves from a point source*, by Professor R. W. McKelvey, University of Colorado, introduced by the Secretary.

The object of the paper is to obtain by a new method, a known expression for a *radiation solution* of the generalized wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} a_i \frac{\partial u}{\partial x_i} + au.$$

The coefficients a_{ij}, a_i, a are variable functions of space, but are constant in time. The matrix (a_{ij})

is positive-definite. [For an exact definition of a radiation solution, see Courant-Hilbert, *Mathematischen Physik II*, Berlin, 1937, p. 453]. The formula in question has been obtained by the methods of Hadamard. [*loc. cit.*, p. 154]. The procedure given here avoids many of the complications of those methods, while preserving the spirit. It consists of a construction process, resembling Hadamard's construction of the fundamental solution. [J. Hadamard, *Lectures on Cauchy's Problem*, New York.]

3. *Coset spaces in topological groups and their relation to the group*, by Mr. D. A. Ford, Graduate Assistant, University of Utah.

A topological group is an abstract group defined on the elements of a topological space where x^{-1} is continuous in x, and the product xy is continuous in x and y simultaneously. If H is a subgroup of a topological group G, then the space of left cosets G/H is a topological space and if H is invariant, G/H is a topological group. The paper deals with the relation of topological properties among the group, subgroup and coset space. For example, if H and G/H are compact, then G is compact.

4. *Geometries based on the undefined terms "sets" and "inclusion,"* by Professor Aboulghassem Zirakzadeh, University of Colorado.

E. V. Huntington, in 1913, introduced a set of axioms for Euclidean Geometry which was based on the undefined terms "set" and "inclusion." Using these undefined terms and changing the given axioms, it is possible to find other geometries, finite and infinite. A minimum set of axioms is given to insure that the resulting geometries are sufficiently regular. The consistency and independence of these axioms is proven.

5. *Variation of parameters in solving systems of difference equations*, by Professor L. C. Barrett and Professor F. E. Dristy, South Dakota School of Mines, presented by Professor Dristy.

The primary purpose of this paper is to illustrate how the method of variation of parameters, so familiar in finding particular integrals of nonhomogeneous linear differential equations, may be extended to determine a particular solution of a nonhomogeneous system of linear difference equations. At the same time a technique is developed for solving such a system which, in contrast to the usual procedure, may be applied directly to the system. Thus, the usual reduction of the system to a single difference equation before solving becomes unnecessary.

6. *Heat conduction with variable thermal conductivity in a sphere*, by Professor Nathan Schwid, University of Wyoming.

When the dependence of the diffusivity coefficient $k/c\rho$ of the heat conduction equation upon the temperature is sufficiently significant to warrant its consideration, the equation becomes nonlinear. The quantity k, the thermal conductivity, is then a function of temperature with $c\rho$, the product of specific heat and density, constant. If we take $k/c\rho$ as $\alpha+\beta u$, where β/α is small, and u is the temperature, an approximation to the temperature in a sphere under simple boundary conditions is obtained which approximates the solution for substantial values of t.

7. *The geometry of* $f(n, \alpha) = \sum e^{ik\alpha}$, $k=0, \cdots, n$, by Professor Emeritus A. J. Kempner, University of Colorado.

An obvious vector construction is combined with the geometrical multiplication in the plane of complex numbers of all points on one curve by all points of a second curve to obtain results of which the following is representative: The "Wertevorrat" (Set of values assumed) of $\sum \sum e^{i(k_1\alpha_1+k_2\alpha_2)} = f(n_1, n_2)$, α_1/π, α_2/π irrational, k_1, k_2 independently over 0, 1, $\cdots$, n_1, and 0, 1, $\cdots$, n_2, respectively, $n_1, n_1 < \infty$ is given by the cardioid $a\rho \sin(\alpha_1/2) \sin(\alpha_2/2) = 1 - \cos(\theta+\alpha_1/2+\alpha_2/2)$. In this cardioid the functional values are distributed everywhere densely.

8. *Mathematics program for outstanding cadets at USAFA*, by Professor W. Milliken, United States Air Force Academy.

Three levels of mathematics have been established at the Air Force Academy. The regular course is the usual two-year engineering mathematics course with spherical trigonometry, some statistics and differential equations added. A cadet may advance from this course to the accelerated course at the beginning of the second semester. The accelerated course covers the same material plus a course in elementary statistics in a year and a half. The super-accelerated course covers this material in one year. Thus, there is extra time available for cadets in the faster programs to take additional advanced mathematics courses or other electives.

9. *Mathematical education in Europe, Britain, and the United States*, by Professor W. W. Rogosinski, King's College, Durham University, England; Visiting Professor, University of Colorado.

An attempt is made to point out and to explain the striking differences in mathematical education, both at high school and university level, as seen in Continental Europe, Britain, and the United States of America. The explanation is sought in a different philosophy of education in general which, in turn, is conditioned by different history and tradition: the scholastic idealism of Europe, its realistic variant in Britain, and the social (and materialistic) trend in American education.

10. *A sequel to Euclid*, by Professor H. S. M. Coxeter, University of Toronto. (Invited Address.)

11. *Coaxal circles and inversion*, by Professor H. S. M. Coxeter, University of Toronto.

Any two given circles belong to a pencil of coaxal circles consisting of all the circles orthogonal to any two circles, α and β, orthogonal to the two given circles. The arbitrariness of α and β is established by inverting the two given circles into straight lines or concentric circles. When inverted with respect to a sphere whose center is outside the plane, two orthogonal pencils of coaxal circles yield sections of a sphere (the inverse of the plane) by pencils of planes through two polar lines. Such a pencil of circles is hyperbolic (*i.e.*, intersecting), parabolic (touching), or elliptic (disjoint) according as the common line of the planes is a secant, a tangent, or an exterior line.

12. *Oscillation and non-oscillation of second order complex differential equations*, by Mr. R. W. Hunt, Graduate Assistant, University of Utah.

The primary object of this paper was to investigate the zeros on $a \leqq x < \infty$ of solutions of the differential equation $(py')' + fy = 0$, with p and f complex-valued continuous functions of the real variable x. By the use of an associated system of two real, second-order equations obtained by writing p, f, and y in polar form, two sufficient conditions for disconjugacy (at most one zero on $a \leqq x < \infty$) of all nontrivial solutions were obtained. Then a special form of this equation, $(y'/q)' + qy = 0$, q complex-valued, was changed to the first order system $y' = q\bar{z}$, $z' = -q\bar{y}$, with solutions $s(x) = s[a, x; q]$ and $c(x) = c[a, x; q]$ corresponding to the boundary conditions $y(a) = 0$, $z(a) = 1$. Finally $s[a, x; q]$ and $c[a, x,; q]$ were shown to have two properties analogous to well-known properties of the real sine and cosine functions; namely, $|s|^2 + |c|^2 = 1$ and, for $k = 1$, $s[a, x; kq] = ks[a, x; q]$, $c[a, x; kq] = c[a, x; q]$.

13. *A multiple integral approach to Taylor's theorem*, by Professor L. C. Barrett and Mr. D. W. Willett, Student, South Dakota School of Mines, presented by Mr. Willett.

This note presents several elementary geometrical considerations, involving lengths, areas, and volumes, which lead quite naturally to a multiple integral approach to Taylor's theorem.

14. *Evaluation of a limit from the theory of heat flow*, by Dr. H. R. Bailey, Mathematician, Ohio Oil Company Research Center, Littleton, Colorado, introduced by the Secretary.

The problem of heat conduction in an infinite homogeneous medium from the surface of a cylinder whose radius is increasing with time is solved by the Green's function method. The solution is obtained as an integral of the form $I=\int_0^t f(t,\tau)d\tau$. A method is given to obtain an explicit evaluation of this integral for $t\to\infty$ for the case of the cylinder radius increasing at a constant velocity. It is shown that the integral can be divided into two parts, $I=\int_0^{t/N} f(t,\tau)d\tau+\int_{t/N}^t f(t,\tau)d\tau$, where the last integral goes to zero as $t\to\infty$ and the integrand in the range $[0, t/N]$ can be replaced by an asymptotic expression which can be integrated explicitly as a function of N. Finally the desired limit is obtained by passing to the limit as $N\to\infty$.

15. *An application of the decomposition of a matrix into principal idempotents*, by Professor D. W. Robinson, Brigham Young University.

As a simple application of the decomposition of a (diagonable) matrix into principal idempotent elements, this note provides a proof of the following well-known result: if the nth derivative of a function f exists at α, then it can be computed as the limit of $h^{-n}\sum_{m=0}^{n}\binom{n}{m}(-1)^m f[\alpha+(n-m)h]$ as h approaches zero.

16. *Families of Sturm-Liouville systems*, by Mr. E. L. Dunn, Colorado State University, introduced by Professor F. M. Stein, Colorado State University.

A family of Sturm-Liouville systems is defined as the collection of all Sturm-Liouville systems whose equations can be obtained by repeatedly differentiating and integrating a Sturm-Liouville equation. For each system, similar boundary conditions apply to the same interval. In this paper the conditions are developed such that a Sturm-Liouville system may generate a family. It is shown that (1) if a Sturm-Liouville system generates a family, the kth derivatives and antiderivatives of its eigenfunctions form orthogonal sets, and (2) if the eigenfunctions of the generating system are not polynomials the sets of eigenvalues for all members of a family are identical. The case when the eigenfunctions are polynomials must be considered separately.

17. *Let's not go off the deep end!* by Professor A. W. Recht, University of Denver.

The general theme is in opposition to the idea of introducing Boolean algebra, sets, and similar types of theoretical mathematics into high school and elementary college courses. We are already teaching too much "gifted" mathematics to the general student, and not teaching successfully the kind of mathematics the 85 per cent or perhaps the 100 per cent, ought to have before they go into the so-called superior pure mathematics. Maybe we have an inferiority complex, and are running away from our real job, which is to bring up all people in our democracy to their full potentialities with a more democratic kind of mathematics.

18. *The work of the Commission on Mathematics*, by Professor Henry Van Engen, University of Wisconsin, Madison.

See this MONTHLY, Report of the May Meeting of the Wisconsin Section.

F. M. CARPENTER, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The twenty-sixth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Carroll College, Waukesha, Wisconsin, on May 3, 1958, Professor R. D. Wagner, Chairman, presiding. Sixty-nine attended the meeting, including forty members of the Association.

At the business meeting of the Section the following officers were elected for the coming year: Chairman, Prof. J. V. Finch, Beloit College, Beloit, Wisconsin; Vice-Chairman, Prof. C. B. Hanneken, Marquette University, Milwaukee, Wisconsin; Secretary-Treasurer, Sister Mary Felice, Mount Mary College, Milwaukee, Wisconsin.

Mr. J. W. Kennedy gave the following report of the 1958 high school mathematics contest: A preliminary contest was held on Feb. 27, in 237 schools in the state, with

9516 students participating. This contest was intended to help teachers choose students for the final contests. Scores on this preliminary test were sent to the committee who then established a score to cut off all but the top 1000 students. Teachers were free, however, to enter any student in the final contest according to their own discretion. The final contest was held at twenty-seven centers throughout Wisconsin, on April 12, with 950 participating from 172 schools.

After a short address of welcome by President Robert D. Steele of Carroll College, the following papers were presented:

1. *On linear inequalities*, by Prof. J. V. Talacko, Marquette University.

As the mathematization of behavioral and empirical sciences continues, we are concerned with solutions of systems of linear inequalities. One class of these problems, useful in Operational Research, Industry and Management, is known as the Linear Programming. The problem was formulated and, by comparison to solutions of a system of linear equations by inverse or unit matrix, the formulation of Simplex Matrix and the Simplex Algorithm was demonstrated as the computational procedure. Finally, a typical maximum mixture problem was formulated, analyzed and solved by simplex.

2. *Integration in finite terms*, by Dr. E. C. Posner, University of Wisconsin, Madison, introduced by Prof. J. V. Finch, Beloit College.

The purpose of this paper was to give some meaning to such oft-repeated statements as "e^x/x cannot be integrated explicitly," and to give some idea of their proof. Working over the complex numbers, we use functions exp and log; start with the algebraic functions, exp and log them, take all rational combinations, in fact, anything satisfying a polynomial equation with these as coefficients. Do this again and again. A function that can be obtained in this way is called elementary. THEOREM. *If* $\int e^{g(x)}y(x)dx$ *is elementary with* $g(x)$, $y(x)$ *rational, then this integral is* $e^{g(x)}w(x)+C$, *for some rational* $w(x)$. COROLLARY. $\int e^x/x\,dx$ *is not elementary.*

3. *Queuing theory and applications*, by Dr. W. A. Golomski, Oscar Mayer & Co., Madison, Wisconsin.

In trying to minimize the waiting time in lines where the input and servicing time is known or can be approximated, a broad theory has been developed in terms of the various cumulative distribution functions involved as well as by differential-difference equations or integral equations. Approximations can be made in many cases in the meat packing industry so that problems of wrapping, hauling, weighing, unloading and loading can be solved by queuing theory techniques. This type of problem requires modifications of known techniques. Examples were given.

4. *A symposium on recent trends in mathematical education*, including:

a. A report on the work of the Commission on Mathematics of the College Entrance Examination Board, by Prof. Henry Van Engen, University of Wisconsin, Madison.

The Commission on Mathematics hopes to redirect the secondary mathematics program by (1) bringing in new mathematical ideas, (2) eliminating ideas which are not in the present stream of mathematical interest, and (3) providing an organization of content based on fundamental mathematical ideas.

In providing for a new high school program, the Commission has kept in mind that calculus must remain as a college subject, but that all work preparatory to a combined analytical geometry and calculus course should be considered as high-school work.

b. Report on the Summer Institute in Social Science for College Teachers of Mathematics, by C. J. Vanderlin, Wisconsin State College, Whitewater.

In addition to lectures on economics, psychology, sociology and social psychology, the members of this institute were given background materials in mathematics, primarily in the fields of

linear inequalities, linear programming, theory of games, and probability. The morning sessions were devoted to these lectures while the afternoon sessions were spent in informal discussion, talks by guest speakers, and "problem generating." An attempt was made to formulate a group of problems, with applications in the social sciences, which could be used in the traditional freshman and sophomore programs. The results were 91 problems presented by the members of the Institute, several of which were read by Mr. Vanderlin.

These papers were followed by lively discussion arising from questions raised particularly by high school teachers present.

SISTER MARY FELICE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Marian College, Indianapolis, November 7, 1958.

IOWA, State University of Iowa, Iowa City, October 17, 1958.

KANSAS

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 6, 1958.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.

MINNESOTA, University of Minnesota, Duluth, Fall, 1958.

MISSOURI, Lindenwood College, St. Charles, Spring, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN, College of the Holy Cross, Worcester, Massachusetts, November 29, 1958.

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City University, October 24, 1958.

PACIFIC NORTHWEST, University of Oregon, Eugene, Oregon, June 19, 1959.

PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.

ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, Spring, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State College, Tempe, Spring, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May, 1959.

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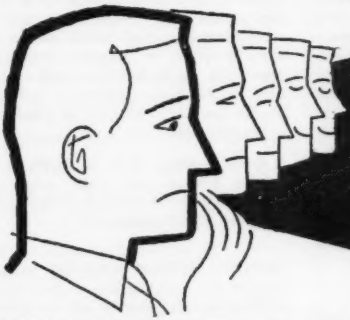

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